Centre for Distance & Online Education
(CDOE)

<u>BACHELOR OF COMMERCE</u>

BCOM 105

BUSINESS MATHEMATICS

Guru Jambheshwar University of Science & Technology, Hisar – 125001

CONTENTS

Lesson. 1 Permutations & Combinations

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Structure:

1.0 Learning Objectives

The following are learning objective of Linear inequalities:

- To define a **permutations & combination** and explain how to calculate one.
- To identify the rule of permutations and combinations.
- To distinguish the similarities and differences between permutations and combinations.
- To correctly choose when to use permutations and combinations in order to solve problems.

1.1 Introduction

Permutations and Combinations

Permutation and combination are all about counting and arrangements made from a certain group of data. The meaning of both these terms is explained here in this article, along with formulas and examples. This is one of the most important topics in the list of mathematics. In this lesson, will discuss all the related concepts with a diverse set of solved questions along with formulas. Moreover, practice questions based on this concept will improve your skills and help you solve any question at your own pace.

What is Permutation?

In mathematics, **permutation relates to the act of arranging all the members of a set into some sequence or order**, or if the set is already ordered, rearranging its elements, a process called permuting. Permutations occur, in more or less prominent ways, in almost every area of mathematics. They often arise when different orderings on certain finite sets are considered.

What is Combination?

The **combination is a way of selecting items from a collection, such that (unlike permutations) the order of selection does not matter**. In smaller cases, it is possible to count the number of combinations. Combination refers to the combination of n things taken k at a time without repetition. To refer to combinations in which repetition is allowed, the terms k-selection or k-combination with repetition are often used.

"My fruit salad is a combination of apples, grapes and bananas" We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.

"The combination to the safe is 472". Now we **do** care about the order. "724" won't work, nor will "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more *precise* language:

- When order doesnot matter it is combination.
- When order does matters it is permutation.

So, we should really call this a "Permutation Lock"!

1.2 Permutation and Combinations

Definition 1.2.1 A Permutation is an ordered **Combination.**

To help you to remember, think "**P**ermutation ... **P**osition"

There are basically two types of permutation:

1. **Repetition is Allowed**: such as the lock above. It could be "333".
2. **No Repetition**: for example the first three people in a running race. You can't be first *and* second.

1. <u>Permutations with Repetition</u>

These are the easiest to calculate.

When a thing has ***n*** different types ... we have ***n*** choices each time!

For example: choosing ***3*** of those things, the permutations are:

$$\mathbf{n \times n \times n}$$

(n multiplied 3 times)

More generally: choosing ***r*** of something that has ***n*** different types, the permutations are:

$$\mathbf{n \times n \times \ldots}\ \textit{(r times)}$$

(In other words, there are **n** possibilities for the first choice, THEN there are **n** possibilites for the second choice, and so on, multplying each time.)

Which is easier to write down using an exponent of **r**:

$$\mathbf{n \times n \times \ldots \text{ (r times)} = n^r.}$$

Example 1.2.2 In the lock above, there are 10 numbers to choose from (0,1,2,3,4,5,6,7,8,9) and we choose 3 of them:

$$\mathbf{10 \times 10 \times ... \text{ (3 times)} = 10^3 = 1{,}000 \text{ permutations}}$$

So, the formula is simply:

$$\mathbf{n^r}$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them, repetition is allowed, and order matters.

2. Permutations without Repetition

In this case, we have to **reduce** the number of available choices each time.

Example 1.2.3 what order could 16 pool balls be in?

After choosing, say, number "14" we can't choose it again.

So, our first choice has 16 possibilites, and our next choice has 15 possibilities, then 14, 13, 12, 11, ... etc. And the total permutations or way of choosing are:

$$16 \times 15 \times 14 \times 13 \times ... = 20{,}922{,}789{,}888{,}000$$

But ma ybe we don't want to choose them all, just 3 of them, and that is then:

$$16 \times 15 \times 14 = 3{,}360$$

In other words, there are 3,360 different ways that 3 pool balls could be arranged out of 16 balls.

Without repetition our choices get reduced each time.

But how do we write that mathematically? Answer: we use the "factorial function"

The **factorial function** (symbol: **!**) just means to multiply a series of descending natural numbers. Examples:

- 4! = 4 × 3 × 2 × 1 = 24
- 7! = 7 × 6 × 5 × 4 × 3 × 2 × 1 = 5,040
- 1! = 1

Note: it is generally agreed that **0! = 1**. It may seem funny that multiplying no numbers together gets us 1, but it helps simplify a lot of equations.

So, when we want to select **all** of the billiard balls the permutations are:

16! = 20,922,789,888,000

But when we want to select just 3 we don't want to multiply after 14. How do we do that? There is a neat trick: we divide by **13!**

16 × 15 × 14 × 13 × 12 ...13 × 12 **...** = 16 × 15 × 14

That was neat. The 13 × 12 × ... etc gets "cancelled out", leaving only 16 × 15 × 14.

The formula is written:

$$\frac{n!}{(n-r)!}$$

where ***n*** is the number of things to choose from,

and we choose ***r*** of them,

no repetitions,

order matters.

Example Our "order of 3 out of 16 pool balls example" is:

16! = 16! = 20,922,789,888,000 = 3,360

$$\frac{}{(16-3)!} \quad \frac{}{13!} \quad \frac{}{6{,}227{,}020{,}800}$$

(which is just the same as: 16 × 15 × 14 = 3,360)

Example 1.2.4 How many ways can first and second place be awarded to 10 people?

$$\frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{3{,}628{,}800}{40{,}320} = 90$$

(which is just the same as: 10 × 9 = 90).

Notation

Instead of writing the whole formula, people use different notations such as these:

$$P(n, r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Example: *P*(10, 2) = 90.

Combinations

Definition 1.2.5 The way of selecting items from a collection where order does not matter is called combination.

There are also two types of combinations (remember the order does not matter now):

1. **Repetition is Allowed**: such as coins in your pocket (5,5,5,10,10)
2. **No Repetition**: such as lottery numbers (2,14,15,27,30,33)

have three scoops. How many variations OK, now we can tackle this one ...

1. Combinations with Repetition

OK, now we can tackle this one ...

let us say there are five flavors of icecream : **banana, chocolate, lemon, strawberry and vanilla**.

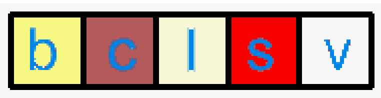
We can have three scoops. How many variations will there be? Let us say there are five flavors of icecream will there be?

Let's use letters for the flavors: {b, c, l, s, v}. Example selections include

- {c, c, c} (3 scoops of chocolate)
- {b, l, v} (one each of banana, lemon and vanilla)
- {b, v, v} (one of banana, two of vanilla)

(And just to be clear: There are **n = 5** things to choose from, and we choose **r = 3** of them. Order does not matter, and we **can** repeat!)

Now, we can't describe directly that how to calculate this, but we can discuss a **special technique** that lets us work it out.



Think about the ice cream being in boxes, we could say "move past the first box, then take 3 scoops, then move along 3 more boxes to the end" and we will have 3 scoops of chocolate!

So it is like we are ordering a robot to get our ice cream, but it doesn't change anything, we still get what we want.

We can write this down as →○○○→→→ (arrow means **move**, circle means **scoop**).

In fact the three examples above can be written like this:

{c, c, c} (3 scoops of chocolate):	→○○○→→→
{b, l, v} (one each of banana, lemon and vanilla):	○→→○→→○

{b, v, v} (one of banana, two of vanilla): ○→→→→○○

OK, so instead of worrying about different flavors, we have a ***simpler*** question: "how many different ways can we arrange arrows and circles?"

Notice that there are always 3 circles (3 scoops of ice cream) and 4 arrows (we need to move 4 times to go from the 1st to 5th container).

So (being general here) there are **r + (n − 1)** positions, and we want to choose ***r*** of them to have circles.

This is like saying "we have **r + (n − 1)** pool balls and want to choose ***r*** of them". In other words it is now like the pool balls question, but with slightly changed numbers. And we can write it like this:

$$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!(n-1)!}$$

where ***n*** is the number of things to choose from,

and we choose ***r*** of them

repetition allowed,

order doesn't matter.

Interestingly, we can look at the arrows instead of the circles, and say "we have ***r*** **+ (n − 1)** positions and want to choose **(n − 1)** of them to have arrows", and the answer is the same:

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1} = \frac{(r+n-1)!}{r!(n-1)!}$$

So, what about our example, what is the answer?

$$\frac{(3+5-1)!}{3!(5-1)!} = \frac{7!}{3!\times 4!} = \frac{5040}{6\times 24} = 35$$

There are 35 ways of having 3 scoops from five flavors of icecream.

2. Combinations without Repetition

This is how **lotteries** work. The numbers are drawn one at a time, and if we have the lucky numbers (no matter what order) we win!

The easiest way to explain it is to:

- assume that the order does matter (i.e., permutations),
- then alter it so the order does **not** matter.

Going back to our pool ball example, let's say we just want to know which 3 pool balls are chosen, not the order.

We already know that 3 out of 16 gave us 3,360 permutations.

But many of those are the same to us now, because we don't care what order!

For example, let us say balls 1, 2 and 3 are chosen. These are the possibilites:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

So, the permutations have 6 times as many possibilites.

In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it. The answer is:

$$3! = 3 \times 2 \times 1 = 6$$

(Another example: 4 things can be placed in $4! = 4 \times 3 \times 2 \times 1 = 24$ different ways, try it for yourself!)

So we adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in their order any more):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

That formula is so important it is often just written in big parentheses like this:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

where ***n*** is the number of things to choose from,

and we choose ***r*** of them,

no repetition,

order doesn't matter.

It is often called "n choose r" (such as "16 choose 3") And is also known as the Binomial Coefficient.

Notation:

As well as the "big parentheses", people also use these notations:

$$C(n, r) = {}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Just remember the formula:

$$\frac{n!}{r!(n-r)!}$$

Example 1.2.6 Pool Balls (without order)

So, our pool ball example (now without order) is:

16! / 3!(16−3)! = *16!* / 3! × 13!

= *20,922,789,888,000* / 6 × 6,227,020,800

= 560

Or we could do it this way:

16×15×14 / 3×2×1 = *3360* / 6 = 560

It is interesting to also note how this formula is nice and **symmetrical**:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r}$$

In other words choosing 3 balls out of 16, or choosing 13 balls out of 16 have the same number of combinations.

$$\frac{16!}{3!(16-3)!} = \frac{16!}{3!.13!} = 560$$

Difference between Permutation and Combination

Permutation	Combination
Arranging people, digits, numbers, alphabets, letters, and colours	Selection of menu, food, clothes, subjects, team.
Picking a team captain, pitcher, and shortstop from a group.	Picking three team members from a group.
Picking two favourite colours, in order, from a colour brochure.	Picking two colours from a colour brochure.
Picking first, second and third place winners.	Picking three winners.

Example 1.2.7 How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that
(i) repetition of the digits is allowed?
(ii) repetition of the digits is not allowed?

Solution. Let the 3-digit number be ABC, where C is at the units place, B at the tens place and A at the hundreds place.

(i) When repetition is allowed:

The number of digits possible at C is 5. As repetition is allowed, the number of digits possible at B and A is also 5 at each.

Therefore, The total number possible 3-digit numbers $=5 \times 5 \times 5 = 125$

(ii) When repetition is not allowed:

The number of digits possible at C is 5. Let's suppose one of 5 digits occupies place C, now as the repletion is not allowed, the possible digits for place B are 4 and similarly there are only 3 possible digits for place A.

Therefore, The total number of possible 3-digit numbers $=5 \times 4 \times 3 = 60$.

Example 1.2.8 How many 3-digits even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

Solutioin. Let the 3-digit number be ABC, where C is at the unit's place, B at the tens place and A at the hundreds place.

As the number has to even, the digits possible at C are 2 or 4 or 6. That is number of possible digits at C is 3.
Now, as the repetition is allowed, the digits possible at B is 6 (any of the 6 is okay). Similarly, at A, also, the number of digits possible is 6.
Therefore, The total number possible 3 digit numbers $=6 \times 6 \times 3 = 108$.

Example 1.2.9 How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

Solution. Let the 4 digit code be 1234.

At the first place, the number of letters possible is 10. Let's suppose any 1 of the ten occupies place 1. Now, as the repetition is not allowed, the number of letters possible at place 2 is 9. Now at 1 and 2, any 2 of the 10 alphabets have been taken. The number of alphabets left for place 3 is 8 and similarly the number of alphabets possible at 4 is 7.

The total number of 4 letter codes $=10 \times 9 \times 8 \times 7 = 5040$.

Example 1.2.10 How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

Solution. Let the five-digit number be ABCDE. Given that first 2 digits of each number is 67. Therefore, the number is 67CDE.

As the repetition is not allowed and 6 and 7 are already taken, the digits available for place C are 0,1,2,3,4,5,8,9. The number of possible digits at place C is 8. Suppose one of them is taken at C, now the digits possible at place D is 7. And similarly, at E the possible digits are 6.

∴T he total five-digit numbers with given conditions=8 × 7 × 6=336.

Example 1.2.11 Evaluate (i) 8 ! (ii) 4 ! – 3 !

Solution. (i) 8! = 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 40320

(ii) 4!-3! = (4 × 3!)-3! = 3!(4-1) = 3 × 2 × 1 × 3 = 18

Example 1.2.12 Evaluate $\frac{n!}{(n - r)!}$,when

(i) n = 6, r = 2

(ii) n = 9, r = 5

Solution. (i) Putting the value of n and r:

$$\frac{6!}{(6-2)!}=\frac{6!}{(4)!}=\frac{6\times5\times4!}{(4)!}=30$$

(ii) Putting the value of n and r:

$$\frac{9!}{(9-5)!}=\frac{9!}{(4)!}=\frac{9\times8\times7\times6\times5\times4!}{(4)!}$$
$$=9\times8\times7\times6\times5=15120$$

Example 1.2.13 How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?

Solution. Total no. of digits possible for choosing = 9

No. of places for which a digit has to be taken = 3

As there is no repetition allowed;

$$\Rightarrow \text{No. of permutations} = \frac{9!}{(9-3)!} = \frac{9!}{(6)!} = \frac{9\times 8\times 7\times 6!}{(6)!}$$
$$= 9\times 8\times 7 = 504$$

Example 1.2.14 How many 4-digit numbers are there with no digit repeated ?

Solution.

In questions like these we need to fill the places that are to be occupied.

To find: Four digit number (digits does not repeat)

Now we will have 4 places where 4 digits are to be put. So, At thousand's place = There are 9 ways as 0 cannot be at thousand's place = 9 ways At hundredth's place = There are 9 digits to be filled as 1 digit is already taken = 9 ways At ten's place = There are now 8 digits to be filled as 2 digits are already taken = 8 ways At unit's place = There are 7 digits that can be filled = 7 ways.

Total Number of ways to fill the four places = 9 × 9 × 8 × 7 = 4536 ways.

So a total of 4536 four digit numbers can be there with no digits repeated.

Example 1.2.15 How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?

Solution. Even number means that last digit should be even,

No. of possible digits at one’s place = 3 (2, 4 and 6)

$$\Rightarrow \text{No. of permutations} = {}^3P_1 = \frac{3!}{(3-1)!} = \frac{3!}{(2)!} = 3.$$

One of digit is taken at one’s place, Number of possible digits available = 5

$$\Rightarrow \text{No. of permutations} = {}^5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{(3)!} = \frac{5\times 4\times 3!}{(3)!} = 20.$$

Therefore, total number of permutations =3 × 20=60.

Example 1.2.16 Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?

Solution. Total no. of digits possible for choosing =5

No. of places for which a digit has to be taken =4

As there is no repetition allowed;

$$\Rightarrow \text{No. of permutations} = \;{}^{5}P_{4} = \frac{5!}{(5-4)!} = \frac{5!}{(1)!} = \frac{5\times4\times3\times2\times1}{1} = 120$$

The number will be even when 2 and 4 are at one's place.

The possibility of (2,4) at one's place = $\frac{2}{5} = 0.4$

Total number of even number = 120 × 0.4 = 48.

Example 1.2.17 From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person cannot hold more than one position?

Solution. Total no. of people in committee = 8

No. of positions to be filled = 2

$$\Rightarrow \text{No. of permutations} = {}^{8}_{2}P = \frac{8!}{(8\text{-}2)!} = \frac{8!}{6!} = \frac{8\times7\times6!}{6!} = 56$$

Example 1.2.18 Find n if ${}^{n-1}P_3 : {}^{n}P_3 = 1 : 9$.

Solution. Here ${}^{-1}P_3 : {}^{n}P_3 = 1 : 9$

$$\frac{{}^{n-1}_{3}P}{{}^{n}_{4}P} = \frac{\dfrac{(n\text{-}1)!}{((n\text{-}1)\text{-}3)!}}{\dfrac{n!}{(n\text{-}4)!}} = \frac{(n\text{-}1)!}{(n\text{-}4)!}.\frac{(n\text{-}4)!}{n!}$$

$$\Rightarrow \qquad = \frac{(n\text{-}1)!}{n(n\text{-}1)!} = \frac{1}{n} = \frac{1}{9}$$

$$\Rightarrow n = 9.$$

Example 1.2.19 How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.

(i) 4 letters are used at a time, (ii) all letters are used at a time,

(iii) all letters are used but first letter is a vowel?

Solution. Total number of letters in MONDAY =6

(i) No. of letters to be used = 4

$\Rightarrow$ No. of permutations = $^6P_4 = \dfrac{6!}{(6\text{-}4)!} = \dfrac{6!}{2!} = \dfrac{6\times5\times4\times3\times2!}{2!} = 360$

(ii) No. of letters to be used = 6

$\Rightarrow$ No. of permutations = $^6P_6 = \dfrac{6!}{(6 - 6)!} = \dfrac{6!}{0!} = \dfrac{6\times5\times4\times3\times2\times1}{1} = 720$

(iii) No. of vowels in MONDAY = 2 (O and A)

$\Rightarrow$ No. of permutations in vowel = $^2P_1 = \dfrac{2!}{(2-1)!} = \dfrac{2!}{(1)!} = 2$.

Now, remaining places = 5

Remaining letters to be used = 5

$\Rightarrow$ No. of permutations = $^5\mathrm{P}_5 = \dfrac{5!}{(5 - 5)!} = \dfrac{5!}{0!} = \dfrac{5\times4\times3\times2\times1}{1} = 120$

Therefore, total number of permutations = 2 × 120 =240.

Example 1.2.20 In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?

Solution. Total number of letters in MISSISSIPPI =11

Letter Number of occurrence

M	1
I	4

S	4
P	2

$\Rightarrow$ Number of permutations = $\dfrac{11!}{1!.4!.4!.2!} = 3465.$

We take that 4 I's come together, and they are treated as 1 letter,

$\therefore$ Total number of letters=11-4+1=8

$\Rightarrow$ Number of permutations = $\dfrac{8!}{1!4!2!} = 840$

Therefore, total number of permutations where four I's don't come together = 34650 - 840 = 33810.

Example 1.2.21 If $^nC_8 = {^nC_2}$, find $^nC_2$.

Solution. Given: $^nC_8 = {^nC_2}$

We know that if $^nC_r = {^nC_p}$ then either r = p or r = n – p

Here $^nC_8 = {^nC_2}$

$\Rightarrow$ 8 = n – 2

$\Rightarrow$ n = 10

Now,

$$\therefore {^nC_2} = {^{10}C_2} = \frac{10!}{2!(10-2)!} \qquad \left(\therefore {^nC_r} = \frac{n!}{r!(n-r)!}\right)$$

$$= \frac{10!}{2!(8)!} = \frac{10\times9\times8!}{2\times1(8)!} = 45.$$

Example 1.2.22 How many chords can be drawn through 21 points on a circle ?

Solution. Given: 21 points on a circle

We know that we require two points on the circle to draw a chord

∴ Number of chords is:

$$\Rightarrow\ {}^{21}C_2 = \frac{21!}{2!(21\text{-}2)!} = \frac{21.20.19!}{2!(19)!} = 210$$

∴ Total number of chords can be drawn are 210

Example 1.2.23 In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

Solution. Given: 5 boys and 4 girls are in total

We can select 3 boys from 5 boys in ${}^5C_3$ ways

Similarly, we can select 3 boys from 54 girls in ${}^4C_3$ ways

∴ No. of ways a team of 3 boys and 3 girls can be selected is ${}^5C_3 \times {}^4C_3$

$$\Rightarrow {}^5C_3 \times {}^4C_3 = {}^5C_3 \times {}^4C_3 = \frac{5!}{3!(5-3)!} \times \frac{4!}{3!(4-3)!} = 10 \times 4$$

$\Rightarrow {}^5C_3 \times {}^4C_3 = 10 \times 4 = 40$

∴ No. of ways a team of 3 boys and 3 girls can be selected is ${}^5C_3 \times {}^4C_3 = 40$ ways.

Example 1.2.24 Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

Solution. Given: 6 red balls, 5 white balls and 5 blue balls

We can select 3 red balls from 6 red balls in ${}^6C_3$ ways

Similarly, We can select 3 white balls from 5 white balls in ${}^5C_3$ ways

Similarly, We can select 3 blue balls from 5 blue balls in ${}^5C_3$ ways

∴ No. of ways of selecting 9 balls is ${}^6C_3 \times {}^5C_3 \times {}^5C_3$

$\Rightarrow {}^6C_3 \times {}^5C_3 \times {}^5C_3 = 20 \times 10 \times 10 = 2000.$

∴ Number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour is ${}^6C_3 \times {}^5C_3 \times {}^5C_3 = 2000$.

Example 1.2.25 In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

Solution. Given: 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers

There are 5 players how can bowl, and we require 4 bowlers in a team of 11

∴ No. Of ways in which bowlers can be selected are: ${}^5C_4$

Now other players left are: 17 – 5(bowlers) = 12

Since we need 11 players in a team and already 4 bowlers are selected, we need to select 7 more players from 12.

∴ No. Of ways we can select these players are: ${}^{12}C_7$

∴ Total number of combinations possible are : ${}^5C_4 \times {}^{12}C_7$

$\Rightarrow {}^5C_4 \times {}^{12}C_7 = 5 \times 792 = 3960.$

∴ Number of ways we can select a team of 11 players where 4 players are bowlers from 17 players are: 3960'

1.3 Check Your Progress

Q.1. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

Q.2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

Q.3. Determine n if

(i) ${}^{2n}C_3 : {}^nC_3$: 12 : 1 (ii) ${}^{2n}C_3 : {}^nC_3 = 11 : 1.$

Q.4. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER?

1.4 Summary

- The number of permutations of n different things taken r at a time, allowing repetitions is n^r.
- The number of permutations of n different things taken all at a time is ${}^nP_n = n!$.
- The number of permutations of n things taken all at a time, in which p are alike of one kind, q are alike of second kind and r are alike of third kind and rest are different is $\frac{n!}{p!q!r!}$.
- Number of permutations of n different things taken r at a time,
  when a particular thing is to be included in each arrangement is $r.{}^{n-1}P_{r-1}$
  when a particular thing is always excluded, then number of arrangements
  $= {}^{n-1}P_r$.
- Number of permutations of n different things taken all at a time, when m specified things always come together is $m!(n+m+1)!$.
- The number of combinations of n different things taken r at a time allowing repetitions is ${}^{n+r-1}C_r$
- The number of ways of dividing n identical things among r persons such that each one gets at least one is ${}^{n-1}C_{r-1}$.
- The total number of combinations of n different objects taken r at a time in which
  (a) m particular objects are excluded = ${}^{n-m}C_r$
  (b) m particular objects are included = ${}^{n-m}C_{r-1}$.

1.5 Keywords

Factorial, Permutation is arrangement and Combination is selection.

1.6 Self-Assessment Test

Q.1. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

Q.2. Find r if (i) $^5P_r = 2\,^6P_{r-1}$ (ii) $^5P_r = {}^6P_{r-1}$. Answer:- r= 3, r = 4.

Q.3. In how many ways can the letters of the word PERMUTATIONS be arranged if the

(i) words start with P and end with S,

(ii) vowels are all together,

(iii) there are always 4 letters between P and S?.

Answer:- (i) 18184400 (ii) 20160 (iii) 254016000.

Q.4. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination. Answer:- 778320.

Q.5. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

Q.6. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

1.7 Answers to check your progress

A.1. The possible outcomes after a coin toss are head and tail.

The number of possible outcomes at each coin toss is 2.

$\therefore$ The total number of possible outcomes after 3 times=$2 \times 2 \times 2$=8.

A.2. Total number of different letters in EQUATION = 8

Number of letters to be used to form a word =8

$\Rightarrow$ No. of permutations = ${}^8_8P = \frac{8!}{(8-8)!} = \frac{8!}{0!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{1} = 40320$.

A.3. Given: ${}^{2n}C_3 : {}^{n}C_3 = 12 : 1$

$${}^{2n}C_3 : {}^{n}C_3 : 12 : 1$$

$$\Rightarrow \frac{{}^{2n}C_3}{{}^{n}C_3} = \frac{12}{1}$$

$$\Rightarrow \frac{\frac{2n!}{3!(2n-3)!}}{\frac{n!}{3!(n-3)!}} = \frac{12}{1}$$

$$\Rightarrow \frac{\frac{2n\,(2n-1)(2n-2)(2n-3)!}{3!(2n-3)!}}{\frac{n\,(n-1)(n-2)(n-3)!}{3!(n-3)!}} = \frac{12}{1}$$

$$\Rightarrow \frac{2n\,(2n-1)(2n-2)}{n\,(n-1)(n-2)} = \frac{12}{1}$$

$$\Rightarrow \frac{2\,(2n-1)2\,(n-1)}{(n-1)(n-2)} = \frac{12}{1}$$

$$\Rightarrow \frac{4\,(2n-1)}{(n-2)} = \frac{12}{1}$$

$\Rightarrow 4\times(2n-1) = 12\times(n-2)$

$\Rightarrow 8n - 4 = 12n - 24$

$\Rightarrow 12n - 8n = 24 - 4$

$\Rightarrow 4n = 20$

$\therefore n = 5$

(ii) On similar Lines.

A.4. The word DAUGHTER has 3 vowels A, E, U and 5 consonants D, G, H, T and R.

The three vowels can be chosen in $^3C_2$ as only two vowels are to be chosen.

Similarly, the five consonants can be chosen in $^5C_3$ ways.

∴ Number of choosing 2 vowels and 5 consonants would be $^3C_2 \times {}^5C_3 = 30$

∴ Total number of ways of is 30

Each of these 5 letters can be arranged in 5 ways to form different words = $^5P_5$

$$= \frac{5!}{(5-5)!} = \frac{5!}{0!} = 120.$$

Total number of words formed would be = 30 × 120 = 3600

1.8 <u>References/ Suggested Readings</u>

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2. Dowling E.T.: Mathematics for Economics; Sihahum Series, McGraw Hill. London.
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Lesson. 2 Binomial Theorem

Course Name: Business Mathematics **Course Code:** BCOM 105

Semester-I **Author:** Dr Vizender Singh

Structure:

2.0 Learning Objectives

The following are learning objective of Binomial Theorem are:

- To extend polynomials and, to identify terms for a given polynomials.
- To generates row of Pascal Triangle.
- To apply in the probability theory of success or failure in a Bernoulli trial.
- To work with combinations.
- Compute binomial coefficients by formula.
- Expand powers of a binomial by Pascal's Triangle and by binomial coefficients.
- Approximate numbers using binomial expansions.

2.1 Introduction

In this lesson we will develop and prove the Binomial Theorem. This theorem tells us about the relationship between two forms of a certain type of polynomial: the factored, binomial form $(x + y)^n$, and the expanded form of this polynomial. To understand the theorem, we will first revisit a topic you have seen previously and we will discuss several other ideas needed to develop the theorem. Then we will prove the theorem using induction. Finally, we will use the theorem to expand polynomials.

2.2 Binomial Theorem

Binomial expression

An algebraic expression consisting of two terms with + *ve* or – *ve* sign between them is called a binomial expression.

For example **:** $(a+b)$, $(2x-3y)$, $\left(\frac{p}{x^2}-\frac{q}{x^4}\right)$, $\left(\frac{1}{x}+\frac{4}{y^3}\right)$ etc.

A **binomial** is a polynomial with two terms

$$\underbrace{3y^2 - 3}_{\text{Example of Binomial}}$$

What happens when we multiply a binomial by itself ... many times?

Example: a + b

a + b is a binomial (the two terms are a and b)

Let us multiply a + b by itself using Polynomial Multiplication :

$$(a + b)(a + b) = a^2 + 2ab + b^2$$

Now take that result and multiply by a + b again:

$$(a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$

And again:

$$(a^3 + 3a^2b + 3ab^2 + b^3)(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The calculations get longer and longer as we go, but there is some kind of pattern developing.

That pattern is summed up by the **Binomial Theorem**:

$$(a + b)^n = \sum_{k=0}^{n} {}^nC_k \, a^{n-k} b^k$$

The Binomial Theorem

Exponents of (a+b)

Now on to the binomial.

We will use the simple binomial a+b, but it could be any binomial.

Let us start with an exponent of **0** and build upwards.

Exponent of 0

When an exponent is 0, we get **1**:

$$(a + b)^0 = 1$$

Exponent of 1

When the exponent is 1, we get the original value, unchanged:

$$(a + b)^1 = a + b$$

Exponent of 2

An exponent of 2 means to multiply by itself (see how to multiply polynomials):

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$

Exponent of 3

For an exponent of 3 just multiply again:

$$(a+b)^3 = (a^2 + 2ab + b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

Now we talk about the pattern.

The Pattern

In the last result we got:

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Now, notice the exponents of a. They start at 3 and go down: 3, 2, 1, 0:

$$\begin{array}{ccccccc} a^3 & + & 3a^2b & + & 3a\,b^2 & + & b^3 \\ 3 & & 2 & & 1 & & 0 \end{array}$$

Likewise the exponents of b go upwards: 0, 1, 2, 3:

$$\begin{array}{ccccccc} a^3 & + & 3a^2b & + & 3a\,b^2 & + & b^3 \\ 0 & & 1 & & 2 & & 3 \end{array}$$

If we number the terms 0 to *n*, we get this:

k = 0	k = 1	k = 2	k = 3
a^3	a^2	a	1
1	b	b^2	b^3

This can be brought together into this:

$$\mathbf{a^{n-k}b^k}$$

Consider an example to see how it works:

Example: When the exponent, *n*, is 3.

The terms are:

k = 0:	k = 1:	k = 2:	k = 3:
$a^{n-k}b^k = a^{3-0}b^0$	$a^{n-k}b^k = a^{3-1}b^1$	$a^{n-k}b^k = a^{3-2}b^2$	$a^{n-k}b^k = a^{3-3}b^3$

$= a^3$	$= a^2b$	$= ab^2$	$= b^3$

Coefficients

So far we have: $a^3 + a^2b + ab^2 + b^3$

But we **really** need: $a^3 + 3a^2b + 3ab^2 + b^3$

We are **missing the numbers** (which are called *coefficients*).

Let's look at **all the results** we got before, from $(a+b)^0$ up to $(a+b)^3$:

$$1$$
$$a + b$$
$$a^2 + 2ab + b^2$$
$$a^3 + 3a^2b + 3ab^2 + b^3$$

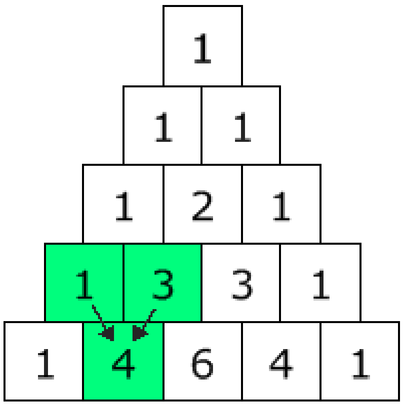
And now look at **just the coefficients** (with a "1" where a coefficient wasn't shown):

$$1$$
$$1a + 1b$$
$$1a^2 + 2ab + 1b^2$$
$$1a^3 + 3a^2b + 3ab^2 + 1b^3$$

They actually make **<u>Pascal's Triangle!</u>**

Each number is just the two numbers above it added together (except for the edges, which are all "1")

(Here we have highlighted that **1+3 = 4)**



Armed with this information let us try something new ... an **exponent of 4**:

a exponents go 4,3,2,1,0: $a^4 + a^3 + a^2 + a + 1$

b exponents go 0,1,2,3,4: $a^4 + a^3b + a^2b^2 + ab^3 + b^4$

coefficients go 1,4,6,4,1: $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

And that is the correct answer (compare to the above results).

Further this pattern can be used for exponents of 5, 6, 7, ... 50, ... 112, ... you name it!

That pattern is the essence of the Binomial Theorem.

The reader can work out $(a + b)^5$ himself.

Answer (hover over): $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

As a Formula

Advancing in lesson we write it all as a formula.

We already have the exponents figured out:

$$\mathbf{a^{n-k}b^k}$$

But how do we write a formula for **"find the coefficient from Pascal's Triangle"** ... ?

Well, there **is** such a formula:

$$^nC_k = \frac{n!}{r!(n - r)!}$$

It is commonly called "n choose k" because it is how many ways to choose k elements from a set of n.

The "!" means "factorial", for example 4! = 4×3×2×1 = 24

You can read more in previous lesson entitled Combinations and Permutations.

And it matches to Pascal's Triangle like this:

(Note how the top row is row zero

and also the leftmost column is zero!)

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

Example 2.2.1 Row 4, term 2 in Pascal's Triangle is "6".

Let's see if the formula works:

$$ {}^{4}C_{2} = \frac{4!}{2!(4-2)!} = \frac{4.3.2!}{2!.2!} = 6 $$

Putting It All Together

The last step is to put all the terms together into one formula.

But we are adding lots of terms together ... can that be done using one formula?

This can be done using Sigma Notation allows us to sum up as many terms as we want:

Now it can all go into one formula:

$$ (x+y)^{n} = \sum_{k=0}^{n} {}^{n}C_{k}\, x^{n-k} y^{k} \qquad (1) $$

The Binomial Theorem

Use It with Example.

For $n = 3$:

$$(a+b)^3 = \sum_{k=0}^{3} {}^3C_k\, a^{3-k} b^k$$

$$= {}^3C_0\, a^{3-0} b^0 + {}^3C_1\, a^{3-1} b^1 + {}^3C_2\, a^{3-2} b^2 + {}^3C_3\, a^{3-3} b^3$$

$$= 1.a^3 + 3.a^2\, b^1 + 3.a\, b^2 + 1.b^3$$

$$= a^3 + a^2\, b^1 + a\, b^2 + b^3$$

BUT ... it is usually **much easier** if we remember the **patterns**:

- The first term's exponents start at **n and go down**
- The second term's exponents start at **0 and go up**
- Coefficients are from Pascal's Triangle, or by calculation using $\frac{n!}{r!(n-r)!}$

Example 2.2.1 What is $(y+5)^4$

Start with exponents:	$\mathbf{y^4 5^0}$	$\mathbf{y^3 5^1}$	$\mathbf{y^2 5^2}$	$\mathbf{y^1 5^3}$	$\mathbf{y^0 5^4}$
Include Coefficients:	$\mathbf{1}y^4 5^0$	$\mathbf{4}y^3 5^1$	$\mathbf{6}y^2 5^2$	$\mathbf{4}y^1 5^3$	$\mathbf{1}y^0 5^4$

Then write down the answer (including all calculations, such as 4×5, 6×5^2, etc):

$$(y+5)^4 = y^4 + 20y^3 + 150y^2 + 500y + 625$$

The coefficient of single term can also be calculated:

Example 2.2.2 What is the coefficient for x^3 in $(2x+4)^8$.

The **exponents** for x^3 are **8 - 5** (=3) for the "2x" and **5** for the "4":

$$(2x)^3 4^5$$

(Why? Because:

2x:	8	7	6	5	4	3	2	1	0
4:	0	1	2	3	4	5	6	7	8
	$(2x)^8 4^0$	$(2x)^7 4^1$	$(2x)^6 4^2$	$(2x)^5 4^3$	$(2x)^4 4^4$	$(2x)^3 4^5$	$(2x)^2 4^6$	$(2x)^1 4^7$	$(2x)^0 4^8$

But we don't need to calculate all the other values if we only want one term.)

And let's not forget "8 choose 5" ... we can use Pascal's Triangle, or calculate directly:

$$^8C_5 = \frac{8!}{5!(8-5)!} = \frac{8.7.6.5!}{5!.3!} = 56$$

And we get:

$$56(2x)^3 4^5$$

Which simplifies to:

$$\mathbf{458752}\ x^3$$

Some important expansions

(1) Replacing y by $-y$ in equation (1) above, we get,

$$(x - y)^n = {^nC_0} x^{n-0} y^0 - {^nC_1} x^{n-1} y^1 + {^nC_2} x^{n-2} y^2 - .... + (-1)^r\ {^nC_r} x^{n-r} y^r + ... + (-1)^n\ {^nC_n} x^0 y^n$$

$$i.e.,\ (x - y)^n = \sum_{r=0}^{n} (-1)^r\ {^nC_r} x^{n-r} y^r$$

The terms in the expansion of $(x-y)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

(2) Replacing x by 1 and y by x in equation (i) we get,

$$(1+x)^n = {^nC_0} x^0 + {^nC_1} x^1 + {^nC_2} x^2 + ... + {^nC_r} x^r + ... + {^nC_n} x^n$$

$$i.e.,\ (1+x)^n = \sum_{r=0}^{n} {^nC_r} x^r$$

This is expansion of $(1+x)^n$ in ascending power of x.

(3) Replacing x by 1 and y by – x in (i) we get,

$$(1-x)^n = {}^nC_0x^0 - {}^nC_1x^1 + {}^nC_2x^2 - ..... +(-1)^r\ {}^nC_rx^r + ...+(-1)^n\ {}^nC_nx^n$$

i.e., $(1-x)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_rx^r$

(4) $(x+y)^n + (x-y)^n = 2[{}^nC_0x^ny^0 + {}^nC_2x^{n-2}y^2 + {}^nC_4x^{n-4}y^4 + ...]$ and

$$(x+y)^n - (x-y)^n = 2[{}^nC_1x^{n-1}y^1 + {}^nC_3x^{n-3}y^3 + {}^nC_5x^{n-5}y^5 + ...]$$

(5) The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is ${}^nC_r$.

(6) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

General term

The general term of the expansion is $(r+1)^{th}$ term usually denoted by T_{r+1} and $T_{r+1} = {}^nC_rx^{n-r}y^r$

- In the binomial expansion of $(x-y)^n, T_{r+1} = (-1)^r\ {}^nC_rx^{n-r}y^r$
- In the binomial expansion of $(1+x)^n, T_{r+1} = {}^nC_rx^r$
- In the binomial expansion of $(1-x)^n, T_{r+1} = (-1)^r\ {}^nC_rx^r$
- In the binomial expansion of $(x+y)^n$, the p^{th} term from the end is $(n-p+2)^{th}$ term from beginning.

Independent term or Constant term

Independent term or constant term of a binomial expansion is the term in which exponent of the variable is zero.

Condition : $(n-r)$ [Power of x] + r [Power of y] = 0, in the expansion of $[x+y]^n$.

Middle term

The middle term depends upon the value of *n*.

(1) **When *n* is even,** then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (odd). So there is only one middle term *i.e.*, $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term. $T_{\left[\frac{n}{2}+1\right]} = {}^nC_{n/2}x^{n/2}y^{n/2}$

(2) **When *n* is odd**, then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (even). So, there are two middle terms *i.e.*, $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ are two middle terms.

$$T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\frac{n-1}{2}}x^{\frac{n+1}{2}}y^{\frac{n-1}{2}} \text{ and } T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\frac{n+1}{2}}x^{\frac{n-1}{2}}y^{\frac{n+1}{2}}$$

- When there are two middle terms in the expansion then their binomial coefficients are equal.
- Binomial coefficient of middle term is the greatest binomial coefficient.

Binomial theorem for any Index

Statement: $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + ...$

$+\frac{n(n-1)......(n-r+1)}{r!}x^r + ...\text{terms up to } \infty$

when *n* is a negative integer or a fraction, where $-1 < x < 1$, otherwise expansion will not be possible.

If first term is not 1, then make first term unity in the following way, $(x+y)^n = x^n\left[1+\frac{y}{x}\right]^n$, if $\left|\frac{y}{x}\right| < 1$.

General term : $T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r$

Some important expansions

i. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ ....+ \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r + ...$

ii. $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - .... \ + \frac{n(n-1)(n-2).....(n-r+1)}{r!}(-x)^r + ...$

iii. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 +$

$$.....+ \frac{n(n+1)......(n+r-1)}{r!}x^r + ...$$

iv. $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 +$

$$.....+ \frac{n(n+1)......(n+r-1)}{r!}(-x)^r + ...$$

v. $(1+x)^{-1} = 1 - x + x^2 - x^3 + ...\infty$

vi. $(1-x)^{-1} = 1 + x + x^2 + x^3 + ...\infty$

vii. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \infty$

viii. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...\infty$

ix. $(1+x)^{-3} = 1 - 3x + 6x^2 - ....\infty$

x. $(1-x)^{-3} = 1 + 3x + 6x^2 + ...\infty$

Example 2.2.3 Find the 6^{th} term in expansion of $\left(2x^2 - \frac{1}{3x^2}\right)^{10}$.

Solution. Applying $T_{r+1} = {}^nC_r x^{n-r} a^r$ for $(x+a)^n$

$$\text{Hence } T_6 = {}^{10}C_5 (2x^2)^5 \left(-\frac{1}{3x^2}\right)^5$$

$$= -\frac{10!}{5!5!} 32 \times \frac{1}{243} = -\frac{896}{27}$$

Example 2.2.4 If the ratio of the coefficient of third and fourth term in the expansion of

$$\left(x - \frac{1}{2x}\right)^n \text{ is } 1 : 2, \text{ then find the value of } n.$$

Solution. $T_3 = {}^nC_2(x)^{n-2}\left(-\frac{1}{2x}\right)^2$ and $T_4 = {}^nC_3(x)^{n-3}\left(-\frac{1}{2x}\right)^3$

But according to the condition,

$$\frac{-n(n-1)\times3\times2\times1\times8}{n(n-1)(n-2)\times2\times1\times4} = \frac{1}{2} \Rightarrow n = -10$$

Example 2.2.5 Find the r^{th} term in the expansion of $(a + 2x)^n$.

Solution. r^{th} term of $(a + 2x)^n$ is ${}^nC_{r-1}(a)^{n-r+1}(2x)^{r-1}$

$$= \frac{n!}{(n-r+1)!(r-1)!}a^{n-r+1}(2x)^{r-1}$$

$$= \frac{n(n-1).....(n-r+2)}{(r-1)!}a^{n-r+1}(2x)^{r-1}$$

Example 2.2.6 If x^4 occurs in the r^{th} term in the expansion of $\left(x^4 + \frac{1}{x^3}\right)^{15}$, then find r.

Solution. $T_r = {}^{15}C_{r-1}(x^4)^{16-r}\left(\frac{1}{x^3}\right)^{r-1} = {}^{15}C_{r-1}x^{67-7r}$

$$\Rightarrow 67-7r = 4 \Rightarrow r = 9.$$

Example 2.2.7 The first 3 terms in the expansion of $(1 + ax)^n$ $(n \neq 0)$ are 1, 6x and $16x^2$. Then find the value of a and n .

Solution. $T_1 = {}^nC_0 = 1$ …..(i)

$T_2 = {}^nC_1ax = 6x$ …..(ii)

$$T_3 = {}^nC_2 (ax)^2 = 16x^2 \qquad \text{.....(iii)}$$

From (ii), $\dfrac{n!}{(n-1)!} a = 6 \Rightarrow na = 6$ .....(iv)

From (iii), $\dfrac{n(n-1)}{2} a^2 = 16$ .....(v)

Using (iv) in (v), we have

$$\Rightarrow \frac{n(n-1)}{2} \left(\frac{6}{n}\right)^2 = 16$$

$$\Rightarrow \frac{n^2(1-\frac{1}{n})}{2} \frac{36}{n^2} = 16$$

$$\Rightarrow 1 - \frac{1}{n} = \frac{8}{9}$$

$$\Rightarrow 1 - \frac{8}{9} = \frac{1}{n}$$

$$\Rightarrow n = 9$$

Using (iv), gives a = $\dfrac{2}{3}$.

Example 2.2.8 If *A* and *B* are the coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then find the relation between A and B.

Solution. Here

$$\frac{\text{Coefficient of } x^n \text{ in expansion of} (1+x)^{2n}}{\text{Coefficient of } x^n \text{ in expansion of } (1+x)^{2n-1}}$$

$$= \frac{{}^{2n}C_n}{{}^{(2n-1)}C_n} = \frac{(2n)!}{n!n!} \times \frac{(n-1)!n!}{(2n-1)!}$$

$$= \frac{(2n)(2n-1)!(n-1)!}{n(n-1)!\,(2n-1)!} = \frac{2n}{n} = 2:1$$

$\Rightarrow \frac{A}{B} = \frac{2}{1} \Rightarrow A = 2B$.

Example 2.2.9 In the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$, find the coefficient of x^4.

Solution. In the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$, the general term is $T_{r+1} = {}^{10}C_r \left(\frac{x}{2}\right)^{10-r} . \left(-\frac{3}{x^2}\right)^r$

$$= {}^{10}C_r (-1)^r . \frac{3^r}{2^{10-r}} x^{10-r-2r}$$

Here, the exponent of x is $10 - 3r = 4 \Rightarrow r = 2$

$$T_{2+1} = {}^{10}C_2 \left(\frac{x}{2}\right)^8 \left(-\frac{3}{x^2}\right)^2 = \frac{10.9}{1.2} . \frac{1}{2^8} .3^2.x^4$$

$$= \frac{405}{256} x^4$$

$\therefore$ The required coefficient $= \frac{405}{256}$.

Example 2.2.10 If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficient of x and x^2 are 3 and – 6 respectively, then find the value of m.

Solution. $(1 + x)^m (1 - x)^n$

$$= \left(1 + mx + \frac{m(m-1)x^2}{2!} + ...\right)\left(1 - nx + \frac{n(n-1)}{2!} x^2 - ....\right)$$

$$= 1 + (m - n)x + \left[\frac{n^2 - n}{2} - mn + \frac{(m^2 - m)}{2}\right] x^2 + ...$$

Given, m – n = 3 or n = m – 3

Hence $\frac{n^2 - n}{2} - mn + \frac{m^2 - m}{2} = -6$

$$\Rightarrow \frac{(m-3)(m-4)}{2} - m(m-3) + \frac{m^2 - m}{2} = -6$$

$$\Rightarrow m^2 - 7m + 12 - 2m^2 + 6m + m^2 - m + 12 = 0$$

$$\Rightarrow -2m + 24 = 0 \Rightarrow m = 12$$

Example 2.2.11 If the coefficient of 4^{th} term in the expansion of $(a+b)^n$ is 56, then find n. **Solution.**

$T_4 = T_{3+1} = {}^nC_3 \; a^{n-3}b^3$

$$\Rightarrow {}^nC_3 = 56 \Rightarrow \frac{n!}{3!\,(n-3)!} = 56$$

$$\Rightarrow n(n-1)(n-2) = 56.6 \Rightarrow n(n-1)(n-2) = 8.7.6$$

$$\Rightarrow n = 8.$$

Example 2.2.12 In the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$, find the term independent of x.

Solution. In the expansion of $\left(\frac{3x^2}{2} + \frac{1}{3x}\right)^9$, the general term is $T_{r+1} = {}^9C_r.\left(\frac{3x^2}{2}\right)^{9-r}\left(-\frac{1}{3x}\right)^r$

$$= {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$$

For the term independent of *x*, $18 - 3r = 0 \Rightarrow r = 6$

This gives the independent term

$$T_{6+1} = {}^9C_6\left(\frac{3}{2}\right)^{9-6}\left(-\frac{1}{3}\right)^6 = {}^9C_3.\frac{1}{6^3}$$

Example 2.2.13 If the middle term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ is $924\,x^6$, then find the valoue of n.

Solution. Since *n* is even therefore $\left(\frac{n}{2}+1\right)^{th}$ term is middle term, hence

$$^{n}C_{n/2}(x^2)^{n/2}\left(\frac{1}{x}\right)^{n/2} = 924x^6$$

$$\Rightarrow\ x^{n/2} = x^6 \ \Rightarrow n = 12.$$

Example 2.2.14 Find the greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$.

Solution. Let (r + 1)th term be the greatest term. Then

$$T_{r+1} = \sqrt{3}.\ ^{20}C_r\left(\frac{1}{\sqrt{3}}\right)^r \text{ and } T_r = \sqrt{3}.\ ^{20}C_{r-1}\left(\frac{1}{\sqrt{3}}\right)^{r-1}$$

Now $\frac{T_{r+1}}{T_r} = \frac{20\text{-}r+1}{r}\left(\frac{1}{\sqrt{3}}\right)$

$$\therefore\ T_{r+1}{}^{3}T_r \Rightarrow 20 - r + 1{}^{3}\sqrt{3}r$$

$$\Rightarrow 21 \geq r(\sqrt{3}+1) \Rightarrow r \leq \frac{21}{\sqrt{3}+1} \Rightarrow r \leq 7.686 \Rightarrow r = 7$$

Hence the greatest term is

$$T_8 = \sqrt{3}\ ^{20}C_7\left(\frac{1}{\sqrt{3}}\right)^7 = \frac{25840}{9}$$

Example 2.2.15 Find the value of $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r}C_n$.

Solution. We have by binomial theorem

$$(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + ... \qquad .....(i)$$

$$\left(1+\frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\frac{1}{x^2} + ... + C_r\frac{1}{x^r} + ... \qquad .....(ii)$$

Multiplying both sides and equating coefficient of x^r in $\frac{1}{x^n}(1 + x)^{2n}$ or the coefficient of x^{n+r} in $(1+ x)^{2n}$ we get the value of required expression

$$= {}^{2n}C_{n+r} = \frac{(2n)!}{(n - r)!(n + r)!}$$

Example 2.2.16 Find the coefficient of x^3 in the expansion of $\frac{(1+3x)^2}{1-2x}$.

Solution. Here

$$\frac{(1 + 3x)^2}{1 - 2x} = (1 + 3x)^2(1 - 2x)^{-1}$$

$$= (1 + 3x)^2\left(1 + 2x + \frac{1.2}{2.1}(-2x)^2 + ...\right)$$

$$= (1+ 6x + 9x^2)(1 + 2x + 4x^2 + 8x^3 + ...)$$

Therefore coefficient of x^3 is $(8 + 24 + 18) = 50$.

Example 2.2.17 If x is positive, then find the first negative term in the expansion of $(1 + x)^{27/5}$.

Solution. Here $(1 + x)^{27/5}$

$$T_{r+1} = \frac{n(n - 1)(n - 2)......(n - r + 1)}{r!}(x)^r$$

For first negative term $n - r + 1 < 0$; $r > \frac{32}{5}$.

$\therefore$ First negative term is 8th term.

Example 2.2.18 If the three consecutive coefficient in the expansion of $(1 + x)^n$ are 28, 56 and 70, then find the value of n .

Solution. Let the consecutive coefficient of $(1+x)^n$ are ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$

By condition, ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 6:33:110$

Now ${}^nC_{r-1} : {}^nC_r = 6 : 33$

$\Rightarrow 2n - 13r + 2 = 0$ …..(i)

and ${}^nC_r : {}^nC_{r+1} = 33 : 110$

$\Rightarrow 3n - 13r - 10 = 0$ …..(ii)

Solving (i) and (ii), we get n = 12 and r = 2.

Example. 2.2.19 Use Binomial theorem to find the value of $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$.

Solution. We have $(x + a)^n + (x - a)^n = 2\ [x^n + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-4} a^4 + {}^nC_6 x^{n-6} a^6 + ...]$

Here, $n = 6,\ x = \sqrt{2},\ a = 1;\ {}^6C_2 = 15,\ {}^6C_4 = 15,\ {}^6C_6 = 1$

$\therefore (\sqrt{2} + 1)^6 (\sqrt{2} - 1)^6 = 2[(\sqrt{2})^6 + 15.(\sqrt{2})^4.1 + 15(\sqrt{2})^2.1 + 1.1]$

$= 2[8 + 15\times4 + 15\times2 + 1] = 198.$

Example. 2.2.20 If $\dfrac{(1 - 3x)^{1/2} + (1 - x)^{5/3}}{\sqrt{4 - x}}$ is approximately equal to $a + bx$ for small values of x, then find the value of (a, b).

Solution. Here $\dfrac{(1 - 3x)^{1/2} + (1 - x)^{5/3}}{\sqrt{4 - x}} = \dfrac{(1 - 3x)^{1/2} + (1 - x)^{5/3}}{2\left[1 - \dfrac{x}{4}\right]^{1/2}}$

$$= \frac{\left[1 + \frac{1}{2}(-3x) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}(-3x)^2 + ...\right] + \left[1 + \frac{5}{3}(-x) + \frac{5}{3}\frac{2}{3}\frac{1}{2}(-x)^2 + ...\right]}{2\left[1 + \frac{1}{2}\left(-\frac{x}{4}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{x}{4}\right)^2 + ...\right]}$$

$$= \frac{\left[1 - \frac{19}{12}x + \frac{53}{144}x^2 - ....\right]}{\left[1 - \frac{x}{2} - \frac{1}{8}x^2 - ....\right]} = 1 - \frac{35}{24}x + ...$$

Neglecting higher powers of x, then

$$a + bx = 1 - \frac{35}{24}x \Rightarrow a = 1, b = -\frac{35}{24}.$$

2.3 Check Your Progress

Q.1. Find the coefficient of *x* in the expansion of $\left(x^2 + \frac{a}{x}\right)^5$.

Q.2. In the expansion of $\left(x - \frac{1}{x}\right)^6$, Find the constant term.

Q.3 Find the coefficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + ...+ (1 + x)^{30}$.

Q.4. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^2$, then find $C_0^2 + C_1^2 + C_2^2 + C_3^2 + ...+ C_n^2$.

Q.5. Find the approximate value of $(7.995)^{1/3}$ correct to four decimal places.

Q.6. Prove that $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + ... = \sqrt{2}.$

Q.7. If the three consecutive coefficients in the expansion of $(1 + x)^n$ are 28, 56 and 70, then find the value of n.

Q.8. The digit in the unit place of the number $(183!) + 3^{183}$.

Q.9. The coefficients of three successive terms in the expansion of $(1 + x)^n$ are 165, 330 and 462 respectively, then find the value of n .

2.4 Summary

- The number of terms in the expansion of $(x + y)^n$ are $(n + 1)$.
- In any term of expansion of $(x + y)^n$, the sum of the exponents of *x* and *y* is always

constant = *n.*

- The binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots$ equidistant from beginning and end are equal *i.e.*, ${}^nC_r = {}^nC_{n-r}$.
- $(x + y)^n =$ Sum of odd terms + Sum of even terms.
- In the expansion of $(x + y)^n, n \in N$ $\frac{T_{r+1}}{T_r} = \left(\frac{n - r + 1}{r}\right)\frac{y}{x}$.
- If the coefficients of p^{th}, q^{th} terms in the expansion of $(1 - x)^n$ are equal, then $p + q = n + 2$.
- The coefficient of x^{n-1} in the expansion of
  $$(x - 1)(x - 2) \ldots\ldots (x - n) = -\frac{n(n + 1)}{2}.$$
- The coefficient of x^{n-1} in the expansion of
  $$(x + 1)(x + 2) \ldots\ldots (x + n) = \frac{n(n + 1)}{2}$$
- For finding the greatest term in the expansion of $(x + y)^n$. we rewrite the expansion in this form $(x + y)^n = x^n \left[1 + \frac{y}{x}\right]^n$. Greatest term in $(x + y)^n = x^n$. Greatest term in $\left(1 + \frac{y}{x}\right)^n$.
- If n is odd, then $(x + y)^n + (x - y)^n$ and $(x + y)^n - (x - y)^n$, both have the same number of terms equal to $\left(\frac{n + 1}{2}\right)$.

- If n is even, then $(x + y)^n + (x - y)^n$ has $\left(\frac{n}{2}+1\right)$ terms and $(x + y)^n - (x - y)^n$ has $\frac{n}{2}$ terms.
- There are infinite number of terms in the expansion of $(1 + x)^n$, when n is a negative integer or a fraction.
- The number of terms in the expansion of $(x_1 + x_2 + ... + x_2)^n = {}^{n+r-1}C_{r-1}$.

2.5 Keywords

Factorial, Pascal Tringle, Arithmatic Progresion, Permutations and Combinations and Geometric Progression.

2.6 Self-Assessment Test

Q.1. In the expansion of $\left(\frac{a}{x}+bx\right)^{12}$, find the coefficient of $x^{-10.}$

Q.2. Find the term independent of x in the expansion of $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^8$.

Q.3. Evaluate $C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n$.

Q.4. If $|x|>1$, then find $(1+x)^{-2}$.

Q.5. Write the first four terms in the expansion of $(1-x)^{3/2}$.

Q.6. Find the the coefficient of x^n in $\frac{(1 + x)^2}{(1 - x)^3}$.

Q.7. Find the coefficient of x^n in the expansion of $\frac{1}{(1 - x)(3 - x)}$.

Q.8. If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then

find $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}$.

Q.9. If the coefficient of the middle term in the expansion of $(1+x)^{2n+2}$ is *p* and the cofficients of middle terms in the expansion of $(1+x)^{2n+1}$ is *q* and *r*, then write the relation between p, q and r.

Q.10. If the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$, then find the value of ab.

Answer :-

1. $^{12}C_1(a)^{11}(b)^1=12a^{11}b$

2. Thus term independent of x = 7.

3. 0

4. $\left[\frac{1}{x^2}-\frac{2}{x^3}+\frac{3}{x^4}-\frac{4}{x^5}+....\right]$

5. $1-\frac{3}{2}x+\frac{3}{8}x^2+\frac{x^3}{16}$

6. $2n^2+2n+1$

7. $\frac{1}{2}\left[1-\frac{1}{3}.\frac{1}{3^n}\right]=\frac{1}{2}\frac{[3^{n+1}-1]}{3^{n+1}}$

8. $2\frac{^nC_{r+1}}{^nC_{r+1}+{}^nC_{r+2}}$ or $\frac{2a_2}{a_2+a_3}$

9. $q+r=p$

10. ab = 1.

2.7 Answers to check your progress

A.1. In the expansion of $\left(x^2+\frac{a}{x}\right)^5$ the general term is $T_{r+1}={}^5C_r(x^2)^{5-r}\left(\frac{a}{x}\right)^r={}^5C_r a^r x^{10-3r}$

Here, exponent of *x* is $10-3r=1\Rightarrow r=3$

$\therefore\ T_{2+1} = {}^5C_3 a^3 x = 10a^3.x$

Hence coefficient of x is $10a^3$.

A.2. In the expansion of $\left(x - \frac{1}{x}\right)^6$, the general term is ${}^6C_r x^{6-r}\left(-\frac{1}{x}\right)^r = {}^6C_r (-1)^r x^{6-2r}$

For term independent of x, $6 - 2r = 0 \Rightarrow r = 3$

Thus the required coefficient $= (-1)^3 . {}^6C_3 = -20$.

A.3. $(1 + x)^{21} + (1 + x)^{22} + ... + (1 + x)^{30}$

$$= (1 + x)^{21}\left[\frac{(1 + x)^{10} - 1}{(1 + x) - 1}\right] = \frac{1}{x}[(1+x)^{31} - (1+x)^{21}]$$

$\therefore$ Coefficient of x^5 in the given expression

$$= \text{Coefficient of } x^5 \text{ in } \left\{\frac{1}{x}[(1 + x)^{31} - (1 + x)^{21}]\right\}$$

$$= \text{Coefficient of } x^6 \text{ in } [(1 + x)^{31} - (1 + x)^{21}]$$

$$= {}^{31}C_6 - {}^{21}C_6.$$

A.4. By Binomial theorem, we have

$$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n \qquad .....(i)$$

$$\text{and } \left(1 + \frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\left(\frac{1}{x}\right)^2 + ... + C_n\left(\frac{1}{x}\right)^n \qquad ....(ii)$$

If we multiply (i) and (ii), we get

$$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2$$

is the term independent of x and hence it is equal to the term independent of x in the product

$$(1+x)^n\left(1+\frac{1}{x}\right)^n \text{ or in } \frac{1}{x^n}(1+x)^{2n} \text{ or term containing } x^n \text{ in } (1+x)^{2n}.$$

Clearly the coefficient of x^n in $(1+x)^{2n}$ is T_{n+1} and equal to ${}^{2n}C_n = \frac{(2n)!}{n!\,n!}$.

A.5. $(7.995)^{1/3} = (8-0.005)^{1/3} = (8)^{1/3}\left[1 - \frac{0.005}{8}\right]^{1/3}$

$$= 2\left[1 - \frac{1}{3}\times\frac{0.005}{8} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2+1}\left(\frac{0.005}{8}\right)^2 + ...\right]$$

$$= 2\left[1 - \frac{0.005}{24} - \frac{\frac{1}{3}\times\frac{1}{3}}{1}\times\frac{(0.005)^2}{8} + ...\right]$$

$$= 2(1-0.000208) = 2\times 0.999792 = 1.9995.$$

A.6. Let the given series be identical with the expansion of $(1+x)^n$ i.e., with $1 + nx + \frac{n(n-1)}{2!}x^2 + ...; |x| < 1$.

Then, $nx = \frac{1}{4}$ and $\frac{n(n-1)}{2}x^2 = \frac{1}{4}.\frac{3}{8} = \frac{3}{32}$

Solving these two equations for n and x. We get $x = -\frac{1}{2}$ and $n = -\frac{1}{2}$.

$\therefore$ Sum of the given series

$$= (1+x)^n = \left(1-\frac{1}{2}\right)^{-1/2} = 2^{1/2} = \sqrt{2}.$$

A.7. Let the three consecutive coefficients be

$^nC_{r-1} = 28, {}^nC_r = 56$ and $^nC_{r+1} = 70$, so that

$$\frac{^nC_r}{^nC_{r-1}} = \frac{n - r + 1}{r} = \frac{56}{28} = 2$$

and $\frac{^nC_{r+1}}{^nC_r} = \frac{n - r}{r + 1} = \frac{70}{56} = \frac{5}{4}$

This gives $n + 1 = 3r$ and $4n - 5 = 9r$

$\therefore \frac{4n - 5}{n + 1} = 3 \Rightarrow n = 8$.

A.8. We know that n! terminates in 0 for $n \geq 5$ and 3^{4n} terminator in 1, $(\because 3^4 = 81)$

$\therefore 3^{180} = (3^4)^{45}$ terminates in 1

Also $3^3 = 27$ terminates in 7

$\therefore 3^{183} = 3^{180}3^3$ terminates in 7.

$\therefore 183! + 3^{183}$ terminates in 7

i.e., the digit in the unit place $= 7$.

A.9. Let the coefficient of three consecutive terms i.e., $(r + 1)^{th}$, $(r + 2)^{th}$, $(r + 3)^{th}$ in expansion of $(1 + x)^n$ are 165,330 and 462 respectively then, coefficient of $(r + 1)^{th}$ term $= {}^nC_r = 165$

Coefficient of $(r + 2)^{th}$ term $= {}^nC_{r+1} = 330$ and

Coefficient of $(r + 3)^{th}$ term $= {}^nC_{r+2} = 462$

$$\therefore \frac{^nC_{r+1}}{^nC_r} = \frac{n - r}{r + 1} = 2$$

or $n - r = 2(r + 1)$ or $r = \frac{1}{3}(n - 2)$

and $\frac{^nC_{r+2}}{^nC_{r+1}} = \frac{n - r - 1}{r + 2} = \frac{231}{165}$

or $165(n - r - 1) = 231(r + 2)$ or $165n - 627 = 396\ r$

or $165n - 627 = 396\times\frac{1}{3}\times(n - 2)$

or $165n - 627 = 132(n - 2)$ or $n = 11$.

2.8 References/ Suggested Readings

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Lesson. 3 Linear Inequalities

Course Name: Business Mathematics **Course Code:** BCOM 105

Semester-I **Author:** Dr Vizender Singh

Structure:

3.0 Learning Objectives

The following are learning objective of Linear inequalities:

- Identify and check solutions to inequalities with two variables.
- Able to draw the graph of each Mathematical statement.
- Graph Solution sets of linear inequalities with two variables.
- **Inequality** tells us about the **relative size** of two values.
- Represent linear inequalities as regions on the coordinate plane.
- Determine if a given point is a solution of a linear inequality.

3.1 Introduction

Linear Inequations: Mathematical expressions help us convert problem statements into entities and thus, help solve them. If the expression equates two expressions or values, then it is called an equation. For e.g. 3x + 5y = 8. On the other hand, if an expression relates two expressions or values with a ‘<’ (less than) sign, ‘>’ (greater than) sign, ‘≤’ (less than or equal) sign or ‘≥’ (greater than or equal) sign, then it is called as an Inequality.

An inequality which involves a linear function is a linear inequality. It looks like a linear equation, except that the ‘=’ sign is replaced by an inequality sign, called linear inequations. In this lesson, we will look at linear inequalities with one or two variables.

3.2 Linear Inequalities

Linear Equations in Two Variables

Definition 3.2.1 An equation that can be put in the form **ax + by + c = 0**, where a, b and c are real numbers and a, b not equal to zero is called a linear equation in two variables namely x and y. The solution for such an equation is a pair of values, one for x and one for y which further makes the two sides of an equation equal.

Solutions to Inequalities with Two Variables

The solution of a linear inequality in two variables like Ax + By > C is an ordered pair (x, y) that produces a true statement when the values of x and y are substituted into the inequality.

Example 3.2.2 Is (1, 2) a solution to the inequality

$$2x + 3y > 1?$$

Solution. Putting x = 1 and y = 2, in the given inequality, we have

$$2\cdot1 + 3\cdot2 > 1$$

$$2 + 5 > 1$$

$$7 > 1$$

which is true, hence the given point is solution of inequality.

How to Graph a Linear Inequality

First, graph the "equals" line, then shade in the correct area.

There are three steps:

- Rearrange the equation so "y" is on the left and everything else on the right.
- Plot the "**y =** " line (make it a solid line for **y ≤** or **y ≥**, and a dashed line for **y <** or **y**>)
- Shade above the line for a "greater than" (**y** > or **y ≥**) or below the line for a "less than" (**y <** or **y** **≤**).

The graph of an inequality in two variables is the set of points that represents all solutions to the inequality. A linear inequality divides the coordinate plane into two halves by a boundary line where one half represents the solutions of the inequality. The boundary line is dashed for > and < and solid for ≤ and ≥. The half-plane that is a solution to the inequality is usually shaded.

Example 3.2.3 Graph the inequality

$$y \geq -x + 1.$$

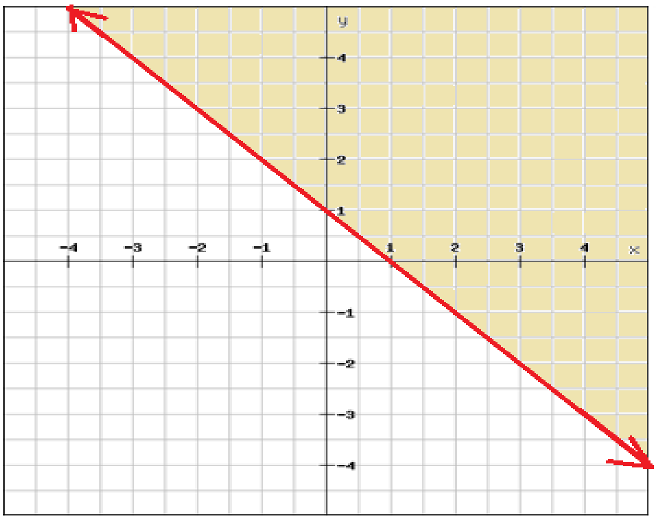
Solution. The given inequality is

$$y \geq -x + 1.$$

To plot the graph changing inequality in to equality

$$y = -x + 1$$

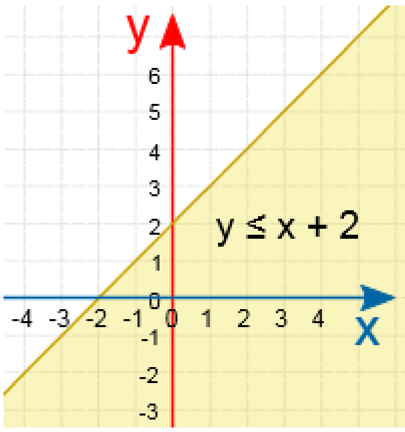
x	1	2	3	0	-1	-2	-3
y	0	-1	-2	1	2	3	4



(Graph)

Example 3.2.4 Graph the inequality $y \leq x + 2$.

Solution. Writing the table of graph on similar lines as in previous example, the required graph is

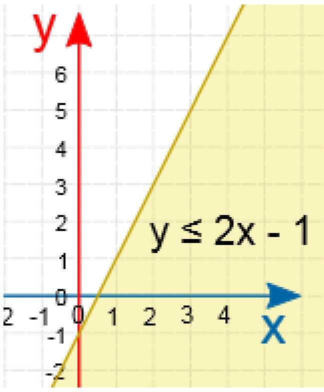


One can see the $y = x + 2$ line, and the shaded area is where y is less than or equal to $x + 2$

Example 3.2.5 Graph the inequality $y \leq 2x - 1$.

Solution. The inequality already has "y" on the left and everything else on the right, so no need to rearrange.

Plot $\mathbf{y = 2x - 1}$ (as a solid line because $y \leq$ includes **equal to**)



The Shaded area is below (because y is **less than** or equal to)

Example 3.2.6 Graph the inequality $2y - x \leq 6$.

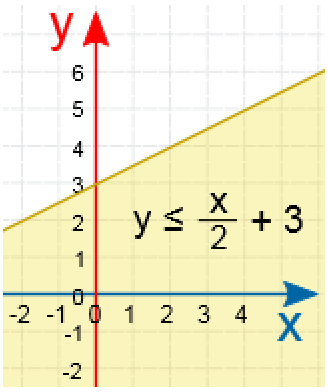
Solution. We will need to rearrange this one so "y" is on its own on the left:

Start with: $2y - x \leq 6$

Add x to both sides: $2y \leq x + 6$

Divide all by 2: $y \leq x/2 + 3$

Now plot **y = x/2 + 3** (as a solid line because y ≤ includes **equal to**)



The Shaded area is below (because y is **less than** or equal to)

Example 3.2.7 Graph the inequality $y/2 + 2 > x$.

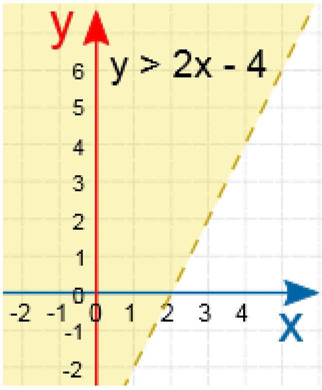
Solution. We will need to rearrange this one so "y" is on its own on the left:

Start with: $y/2 + 2 > x$

Subtract 2 from both sides: $y/2 > x - 2$

Multiply all by 2: $y > 2x - 4$

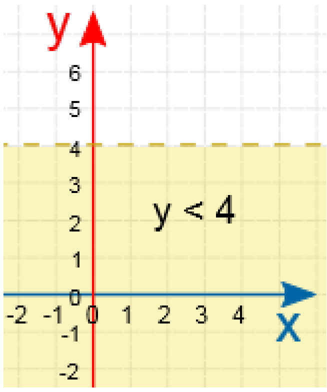
Now plot **y = 2x − 4** (as a dashed line because y> does not include equals to)



The dashed line shows that the inequality does **not** include the line **y = 2x - 4**.

Example 3.2.8 Graph the inequality $y < 4$ and $x \geq 1$.

Solution. For inequality $y < 4$ the graph is

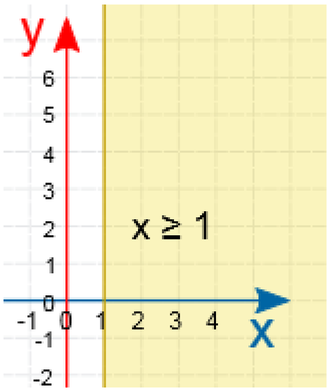


This shows where y is less than 4

(from, but not including, the line **y = 4** on down)

Notice that we have a dashed line to show that it does not include where y = 4.

For inequality $x \geq 1$the graph is

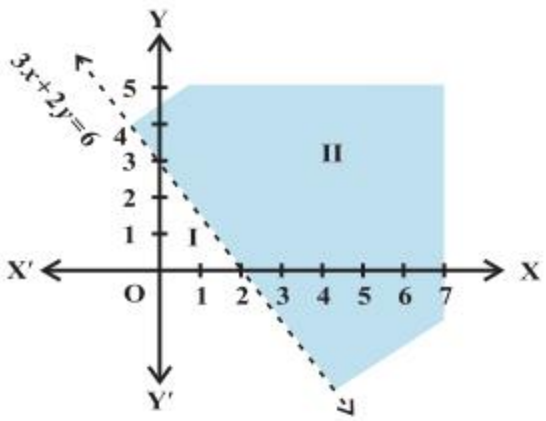


This one doesn't even have y in it!

It has the line **x=1**, and is shaded for all values of x greater than (or equal to) 1

Example 3.2.9 Solve $3x + 2y > 6$ graphically in a two-dimensional plane.

Solution. To solve the inequality, let's plot a graph of the equation $3x + 2y = 6$ as shown below:



This line divides the plane into half-planes I and II. Next, we select a point (0, 0) and determine if it satisfies the given inequality. Hence, we have

$3x + 2y > 6$

$\Rightarrow 3(0) + 2(0) > 6$

$\Rightarrow 0 > 6$ which is FALSE.

Since (0, 0) lies in the half-plane I and it does not satisfy the inequality, half-plane I, is not the solution. Also, the inequality given is a 'Strict inequality'. Hence, any point on the line represented by $3x + 2y = 0$ does not satisfy the inequality either. Therefore, the solution of the inequality is the shaded region in the diagram above.

Example 3.2.10 Solve $3x + 2y > 6$ graphically in a two-dimensional plane.

Solution. Given in equality is : $x + y < 5$

Consider: $x + y = 5$

x	0	5
y	5	0

Now draw a dotted line $x + y = 5$ in the graph ($\because$ $x + y = 5$ is excluded in the given question)

Now Consider $x + y < 5$

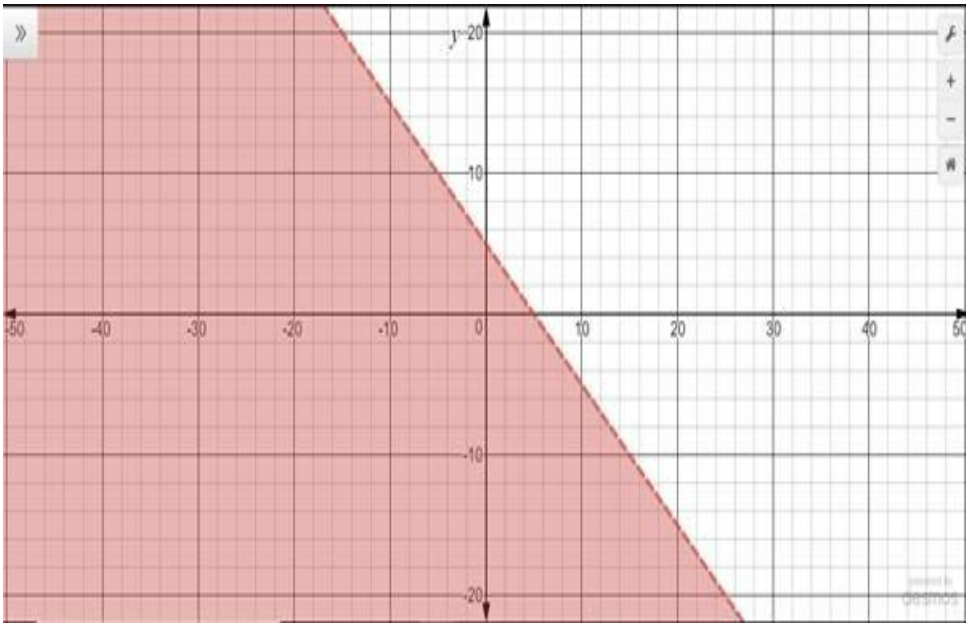
Select a point (0,0)

$\Rightarrow 0 + 0 < 5$

$\Rightarrow 0 < 5$ (this is true)

$\therefore$ Solution region of the given inequality is below the line $x + y = 5$. (That is origin is included in the region)

The graph is as follows:



Example 3.2.11 Solve the following inequalities graphically in two-dimensional plane: $2x + y \geq 6$.

Solution. Given inequality is: $2x + y \geq 6$

Consider: $2x + y = 6$

x	0	3
y	6	0

Now draw a solid line $2x + y = 6$ in the graph ($\because 2x + y = 6$ is included in the given question)

Now Consider $2x + y \geq 6$

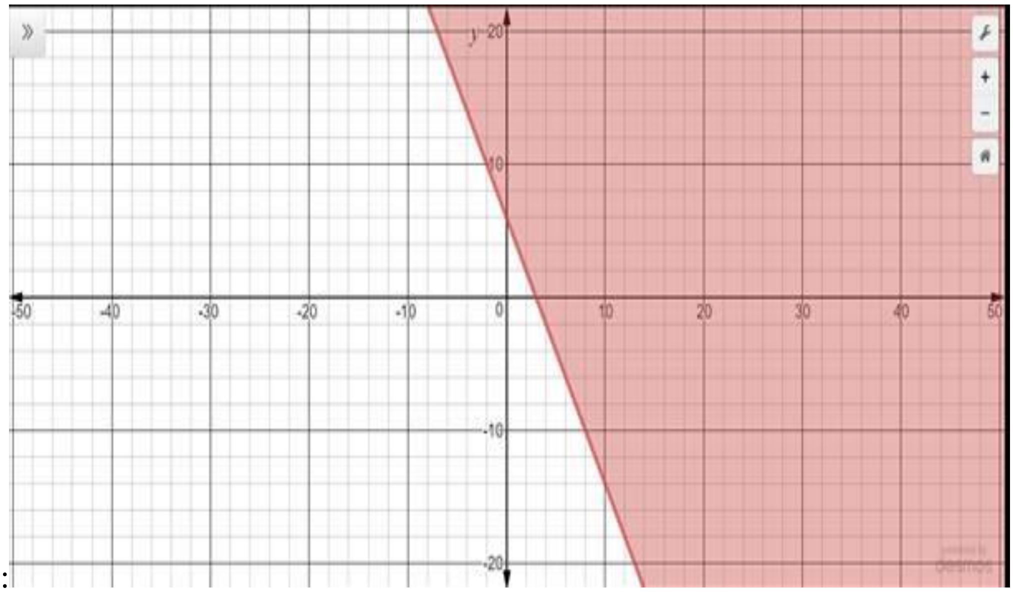
Select a point (0,0)

$\Rightarrow 2 \times (0) + 0 \geq 6$

$\Rightarrow 0 \geq 5$ (this is false)

$\therefore$ Solution region of the given inequality is above the line $2x + y = 6$. (Away from the origin)

The graph is as follows:



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Example 3.2.12 Solve the following inequalities graphically in two-dimensional plane: $3x + 4y \leq 12$.

Solution. Given: $3x + 4y \leq 12$

Consider: $3x + 4y = 12$

x	0	4
y	3	0

Now draw a solid line $3x + 4y = 12$ in the graph ($\because 3x + 4y = 12$ is included in the given question)

Now Consider $3x + 4y \leq 12$

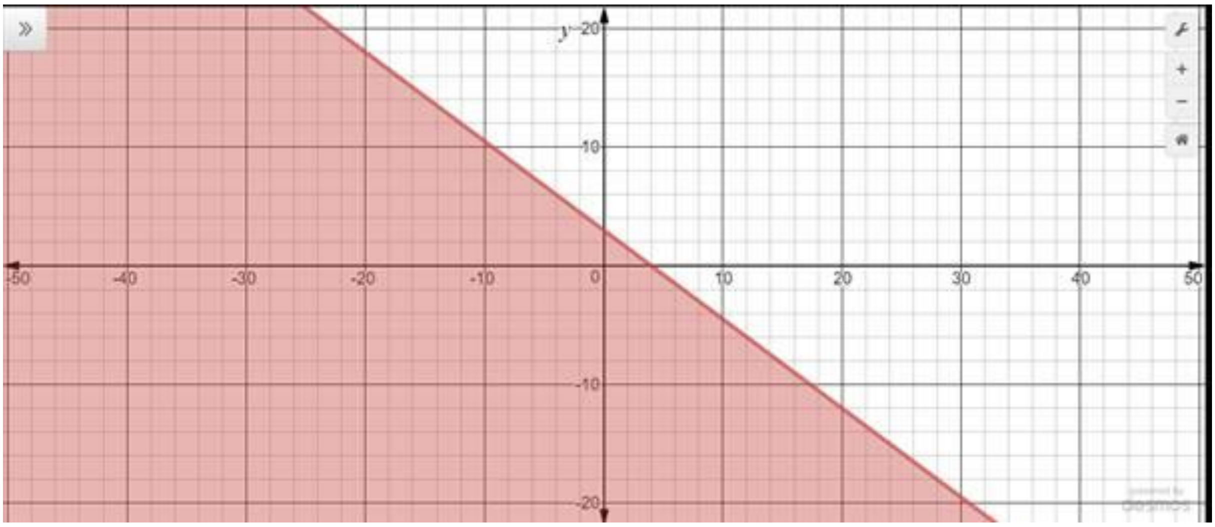
Select a point (0,0)

$\Rightarrow 3 \times (0) + 4 \times (0) \leq 12$

$\Rightarrow 0 \leq 12$ (this is true)

$\therefore$ Solution region of the given inequality is below the line $3x + 4y = 12$. (That is origin is included in the region)

The graph is as follows:



Example 3.2.13 Solve the following inequalities graphically in two-dimensional plane:

$y + 8 \geq 2x.$

Solution. Given: $y + 8 \geq 2x$

Consider: $y + 8 = 2x$

x	0	4
y	-8	0

Now draw a solid line $y + 8 = 2x$ in the graph ($\because y + 8 = 2x$ is included in the given question)

Now Consider $y + 8 \geq 2x$

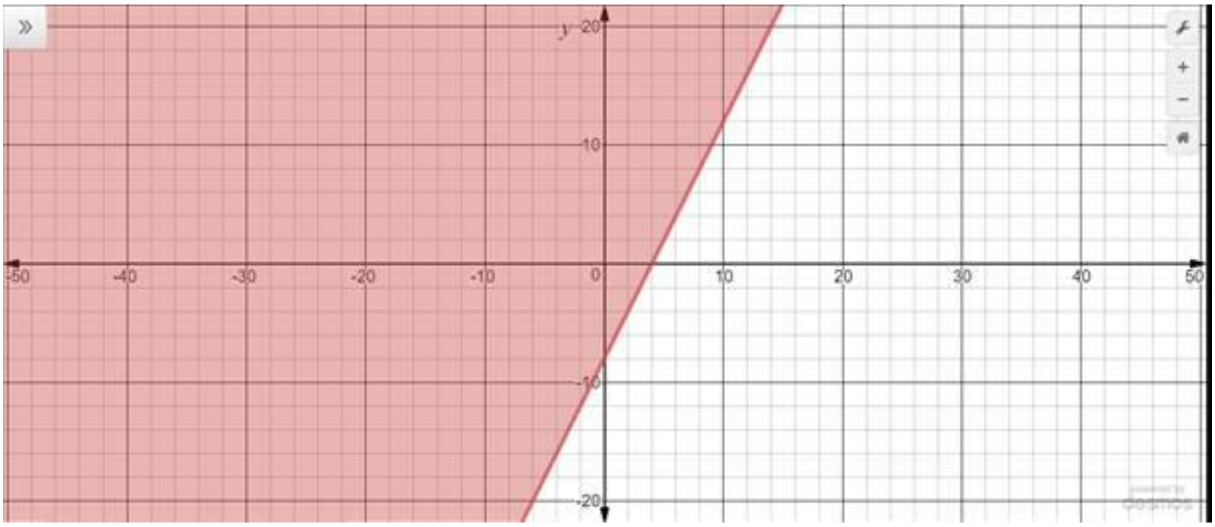
Select a point (0,0)

$\Rightarrow (0) + 8 \geq 2 \times (0)$

$\Rightarrow 0 \leq 8$ (this is true)

$\therefore$ Solution region of the given inequality is above the line $y + 8 = 2x$. (That is origin is included in the region)

The graph is as follows:



Example 3.2.14 Solve the following system of linear inequalities in two variables graphically.

$$x + y \geq 5$$

$$x - y \leq 3$$

Solution. To begin with, let's draw a graph of the equation x + y = 5. Now, we determine if the point (0, 0), which is lying in the half-plane I, satisfies the inequality 1. We have,

$$x + y \geq 5$$
$$\Rightarrow 0 + 0 \geq 5$$

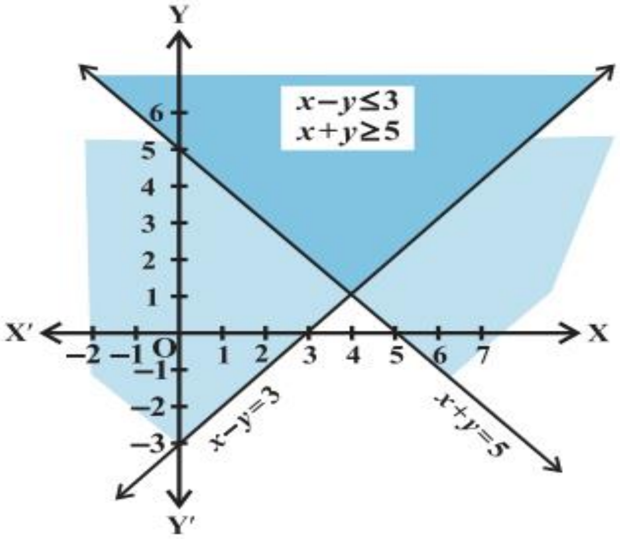
Or, $0 \geq 5$, which is FALSE. Also, being a 'Slack inequality, the points on the line represented by x + y = 5 satisfy the inequality 1. Hence, the solution lies in the half-plane II and includes the line.

Next, let's draw a graph of the equation x – y = 3 on the same set of axes. Now, we determine if the point (0, 0), which is lying in the half-plane II, satisfies the inequality 2. We have,

$x - y \leq 3$

$\Rightarrow 0 - 0 \leq 3$

Or, $0 \leq 3$ which is TRUE. Also, being a 'Slack inequality, the points on the line represented by x – y = 3, satisfy the inequality 2. Hence, the solution lies in the half-plane II and includes the line. Look at the diagram below:



The solution of the system of linear inequalities in two variables x + y ≥ 5 and x – y ≤ 3 is the region common to the two shaded regions as shown above.

Example 3.2.15 Solve the following system of inequalities graphically: x ≥ 3, y ≥ 2.

Solution. Given x ≥ 3………1

y ≥ 2……………2

Since x ≥ 3 means for any value of y the equation will be unaffected so similarly for y ≥ 2, for any value of x the equation will be unaffected.

Now putting x = 0 in the 1

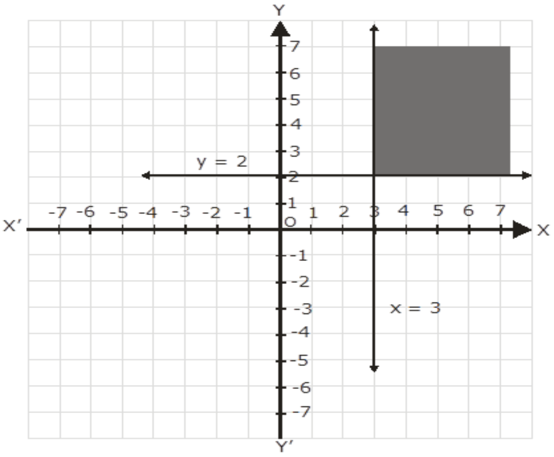
0 ≥ 3 which is not true

Putting y = 0 in 2

0 ≥ 2 which is not true again

This implies the origin doesn't satisfy in the given inequalities. The region to be included will be on the right side of the two equalities drawn on the graphs.

The shaded region is the desired region.



inequality.

⇒ $0 \geq 2$ which is not true, hence origin is not included in the solution of the

which is covered by all the given three inequalities at the same time satisfying all the given inequality.

The region to be included in the solution would be towards the left of the equality $y \geq 2$

The shaded region in the graph will give the answer to the required inequalities as it is the region conditions.

Example 3.2.16 Solve the following system of inequalities graphically: $3x + 2y \leq 12$, $x \geq 1$,

$y \geq 2$.

Solution. Given $3x + 2y \leq 12$

Solving for the value of x and y by putting $x = 0$ and $y = 0$ one by one

We get

$y = 6$ and $x = 4$

So the points are (0,6) and (4,0)

Now checking for (0,0)

$0 \leq 12$ which is also true,

Hence the origin lies in the plane and the required area is toward the left of the equation.

Now checking for $x \geq 1$,

The value of x would be unaffected by any value of y

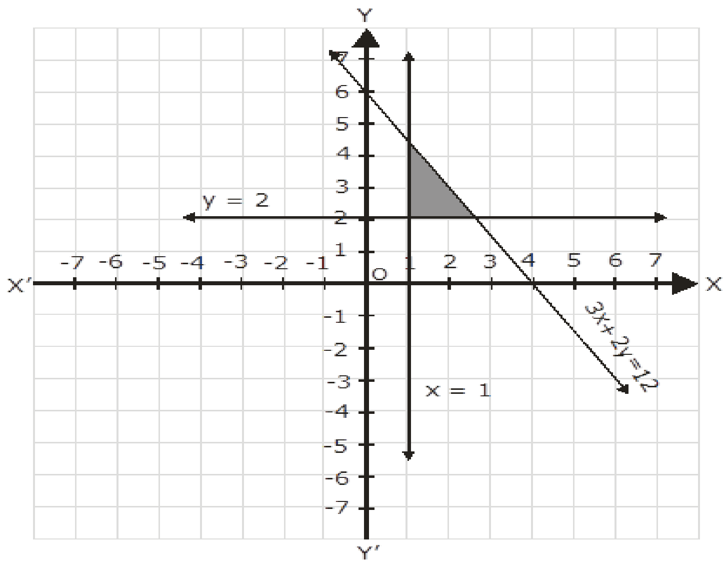
The origin would not lie on the plane

$\Rightarrow 0 \geq 1$ which is not true

The required area to be included would be on the left of the graph $x \geq 1$

Similarly, for $y \geq 2$

Value of y will be unaffected by any value of x in the given equality. Also, the origin doesn't satisfy the given



Example 3.2.17 Solve the following system of inequalities graphically: $2x + y \geq 6$,

$3x + 4y \leq 12$.

Solution. Given $2x + y \geq 6$……………1

$3x + 4y \leq 12$ …………….2

In (1) $2x + y \geq 6$

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 6 and x = 3

So the point for the (0,6) and (3,0)

Now checking for (0,0)

$0 \geq 6$ which is not true, hence the origin does not lies in the solution of the equality. The required region is on the right side of the graph.

Checking for $3x + 4y \leq 12$

Putting value of x= 0 and y = 0 one by one in equation

We get y = 3, x = 4

The points are (0, 3) , (4, 0)

Now checking for origin (0, 0)

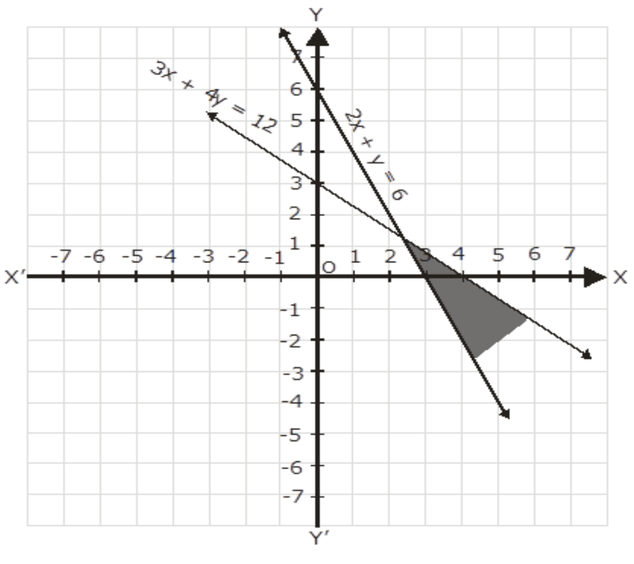
$0 \leq 12$ which is true,

so the origin lies in solution of the equation.

The region on the right of the equation is the region required.

The solution is the region which is common to the graphs of both the inequalities.

The shaded region is the required region.



Example 3.2.18 Solve the following system of inequalities graphically: $x + y \geq 4$, $2x - y < 0$.

Solution. Solving for $x + y \geq 4$

Putting value of $x = 0$ and $y = 0$ in equation one by one, we get value of

$y = 4$ and $x = 4$

The points for the line are (0, 4) and (4, 0)

Checking for the origin (0, 0)

$0 \geq 4$

This is not true,

So the origin would not lie in the solution area. The required region would be on the right of line`s graph.

$2x - y < 0$

Putting value of $x = 0$ and $y = 0$ in equation one by one, we get value of

$y = 0$ and $x = 0$

Putting $x = 1$ we get $y = 2$

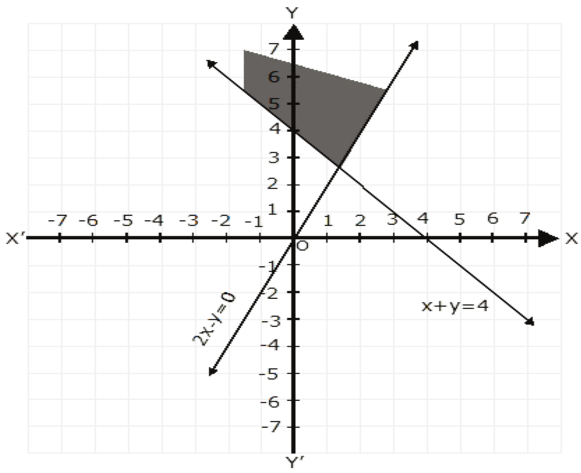
So the points for the given inequality are (0, 0) and (1,2)

Now that the origin lies on the given equation we will check for (4,0) point to check which side of the line`s graph will be included in the solution.

$\Rightarrow 8 < 0$ which is not true, hence the required region would be on the left side of the

line $2x-y < 0$

The shaded region is the required solution of the inequalities.



Example 3.2.19 Solve the following system of inequalities graphically: 2x – y >1,

x – 2y < – 1.

Solution. Given 2x – y >1………………1

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = -1 and x = 1/2 = 0.5

The points are (0,-1) and (0.5 , 0)

Checking for the origin, putting (0,0)

0 >1 , which is false

Hence the origin does not lie in the solution region. The required region would be on the right of the line`s graph.

x – 2y < – 1…………2

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 1/2 = 0.5 and x = -1

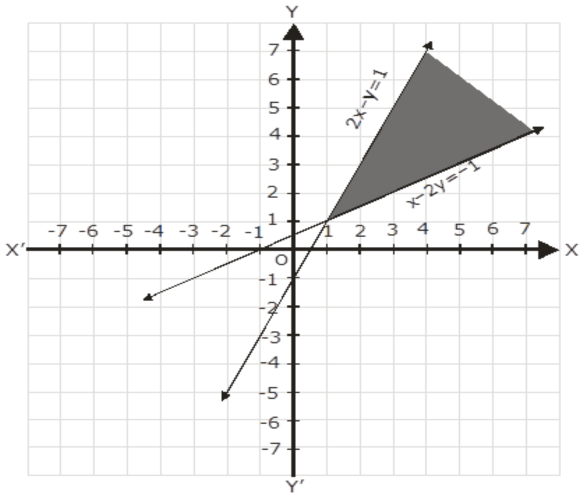
The required points are (0, 0.5) and (-1, 0)

Now checking for the origin, (0,0)

0 < -1 which is false

Hence the origin does not lies in the solution area, the required area would be on the left side of the line`s graph.

∴ the shaded area is the required solution of the given inequalities.



3.3 Check Your Progress

Q.1. Check for the points (1, 3) and (3,-2) if these ordered pairs are in the solution set of the inequality $y \geq 2x - 1$ or not.

Q.2. Solve graphically $y < 2x + 1$.

Q.3. Solve the following inequalities graphically in two-dimensional plane:

$x - y \leq 2$.

Q.4. Solve the following inequalities graphically in two-dimensional plane:

$2x - 3y > 6$.

Q.5. Solve the following inequalities graphically in two-dimensional plane:

$-3x + 2y \geq -6$.

Q.6. Draw the graph of following system of inequalities: $x + y \leq 6$, $x + y \geq 4$.

Q.7. Draw the graph of following system of inequalities: $2x + y \geq 8$, $x + 2y \geq 10$.

Q.8. Draw the graph of following system of inequalities: $x + y \leq 9$, $y > x$, $x \geq 0$.

Q.9 Draw the graph of following system of inequalities: $5x + 4y \leq 20$, $x \geq 1$, $y \geq 2$.

Q.10 Solve the following system of inequalities graphically: $3x + 4y \leq 60$, $x + 3y \leq 30$, $x \geq 0$ $y \geq 0$

3.4 Summary

Linear inequalities with two variables have infinitely many ordered pair solutions, which can be graphed by shading in the appropriate half of a rectangular coordinate plane.

To graph the solution set of an inequality with two variables, first graph the boundary with a dashed or solid line depending on the inequality. If given a strict inequality, use a dashed line for the boundary. If given an inclusive inequality, use a solid line. Next, choose a test point not on the boundary. If the test point solves the inequality, then shade the region that contains it; otherwise, shade the opposite side.

Check your answer by testing points in and out of the shading region to verify that they solve the inequality or not.

3.5 Keywords

Linear equation in two variables; Graph Plotting; Coordinate Ponts.

3.6 Self-Assessment Test

Q.1. Is the order pair solution of given inequality?

(i) $5x - y > -2$; (-3, -5) (ii) $4x - y < -8$; (-3, -10)

(ii) $6x - 15y > -1$; (1/2, -1/3) (iv) $x - 2y \geq 2$; (2/3, -5/6).

Answers. (i) No (ii) No (iii) No (iv) No.

Q.2. Find the graphical solution of linear inequality.

(i) $y \geq - 2/3\, x + 3$ (ii) $2x + 3y \leq 18$ (ii) $6x - 5y > 30$

(iv) $4x - 4y < 0$ (v) $x + > 0$ (vi) $y \leq -2$

(vii) $x < -2$ (viii) $5x \leq -4\,y - 12$ (ix) $4y + 2 < 3x$

(x) $5 \geq 3x - 15\,y$.

Q.3. Solve the following inequalities graphically in two-dimensional plane:

$y - 5x < 30$

Q.4. Solve the following inequalities graphically in two-dimensional plane:

$y < -2$

Q.5. Solve the following inequalities graphically in two-dimensional plane:

$x > -3$

Q.6. Solve the following system of inequalities graphically: $2x + y \geq 4$, $x + y \leq 3$, $2x - 3y \leq 6$

Q.7. Solve the following system of inequalities graphically: $x - 2y \leq 3$, $3x + 4y \geq 12$,

$x \geq 0$, $y \geq 1$.

Q.8. Solve the following system of inequalities graphically: $4x + 3y \leq 60$, $y \geq 2x$, $x \geq 3$,

$x, y \geq 0$.

Q.9. Solve the following system of inequalities graphically: $3x + 2y \leq 150$, $x + 4y \leq 80$,

$x \leq 15$, $y \geq 0$, $x \geq 0$

Q.10. Solve the following system of inequalities graphically: $x + 2y \leq 10$, $x + y \geq 1$,

$x - y \leq 0$, $x \geq 0$, $y \geq 0$.

3.7 Answers to check your progress

A.1. The given inequality is; $y \geq 2x-1$

1. Let check for (1, 3) first.

Substitute $x = 1$ and $y = 3$

$3 \geq 2(1) - 1$

$3 \geq 1$

This is a true statement, so this ordered pair comes in the solution region.

2. Let check for (3,-2) now

Substitute $x=3$ and $y = -2$

$-2 \geq 2(3) - 1$

$-2 \geq 5$

This is a false statement, so this ordered pair is not comes in the solution region.

If you know the solution region then you can check it by plotting the point on the graph.

A.2. First we will find the solution of the equation by assuming x = 0 first and then x = 1.y = 2x +1

Let $x = 0$

$y = 2(0) + 1$

$y = 1$

Let $x = 1$

$y = 2(1) + 1$

$y = 3$

So the coordinates of the line are (0, 1) and (1, 3)

2. Joining these two points we will get the line of equation $y = 2x + 1$.

3. As the sign of inequality is < that is, the strict inequality so the boundary line will be a dashed line.

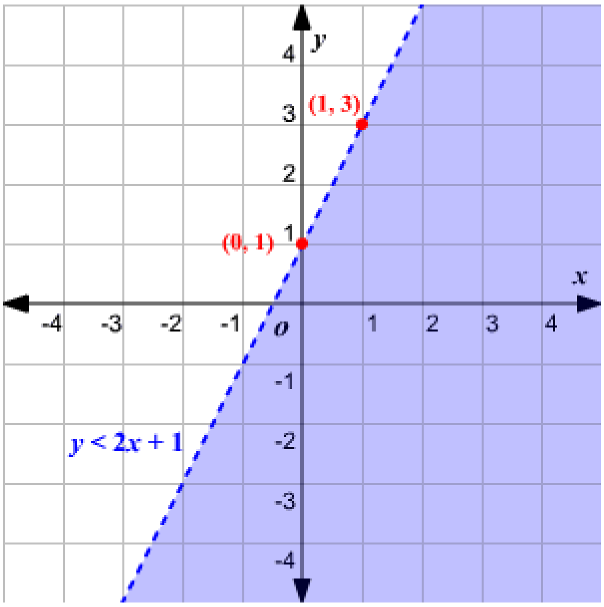
4. Now we will check for the solution region by substituting x = 0 and y = 0.

$y < 2x + 1$

$0 < 2(0) + 1$

This is a true statement so the solution region will be on the half-plane which contains (0, 0) coordinates.

5. So we will shade the lower half plane that is, below the boundary line.



A.3. Given: $x - y \leq 2$

Consider: $x - y = 2$

x	0	2
y	-2	0

Now draw a solid line $x - y = 2$ in the graph ($\because$ $x - y = 2$ is included in the given question)

Now Consider $x - y \leq 2$

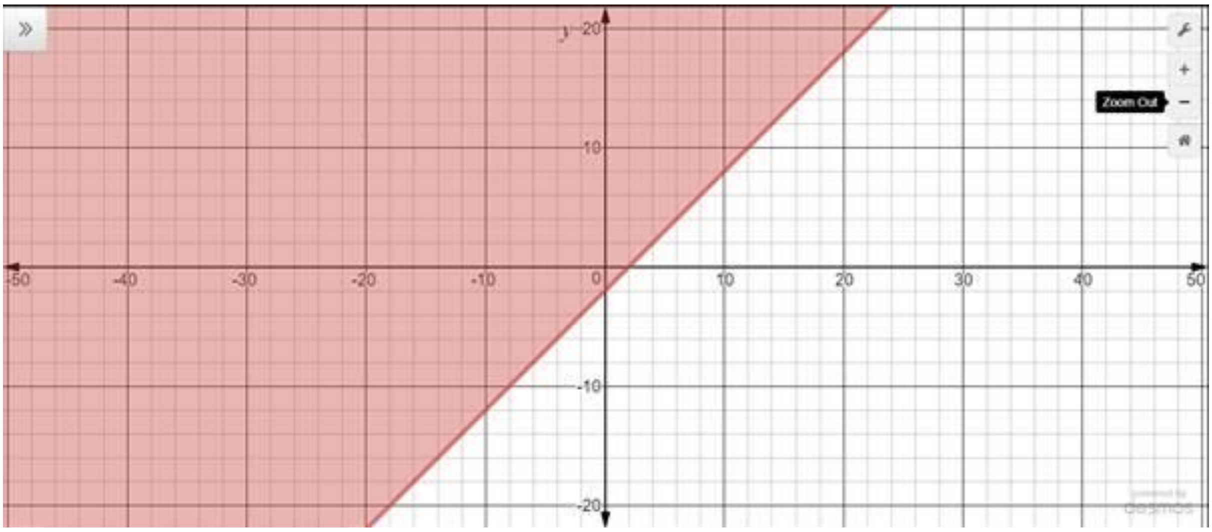
Select a point (0,0)

$\Rightarrow$ (0) – (0) $\leq$ 2

$\Rightarrow$ 0 $\leq$ 12 (this is true)

$\therefore$ Solution region of the given inequality is above the line $x - y = 2$. (That is origin is included in the region)

The graph is as follows:



A.4. Given: 2x – 3y > 6

Consider: 2x – 3y = 6

x	0	3
y	-2	0

Now draw a dotted line 2x – 3y = 6 in the graph (∵2x – 3y = 6 is excluded in the given question)

Now Consider 2x – 3y > 6

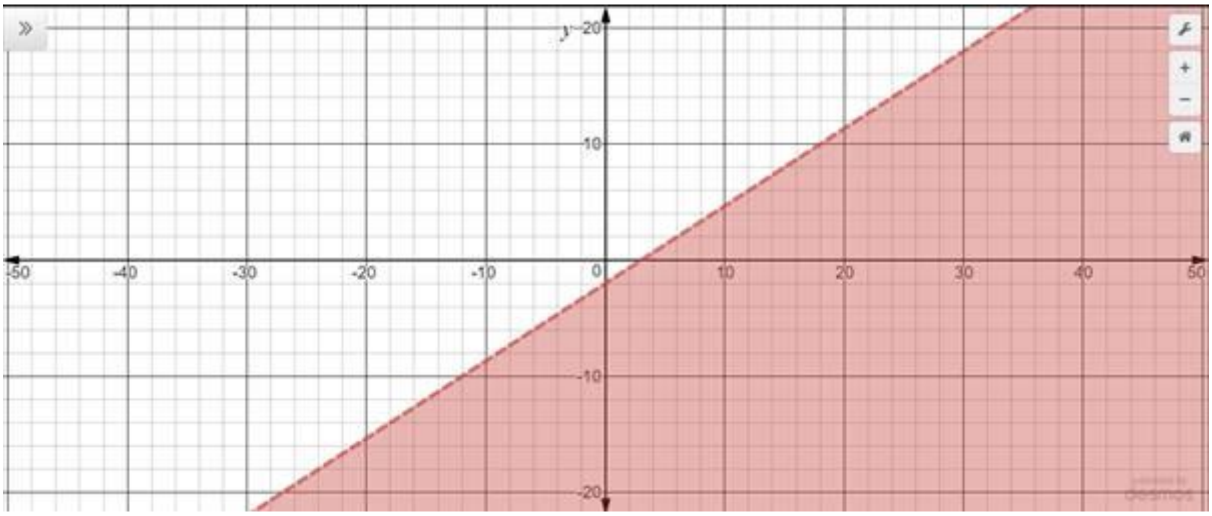
Select a point (0,0)

⇒ 2 × (0) – 3 × (0) > 6

⇒ 0>5 (this is false)

∴ Solution region of the given inequality is below the line 2x – 3y > 6. (Away from the origin)

The graph is as follows:



A.5. Given: – 3x + 2y ≥ – 6

Consider: – 3x + 2y = – 6

x	0	2
y	-3	0

Now draw a solid line – 3x + 2y = – 6 in the graph (∵– 3x + 2y = – 6 is included in the given question)

Now Consider – 3x + 2y ≥ – 6

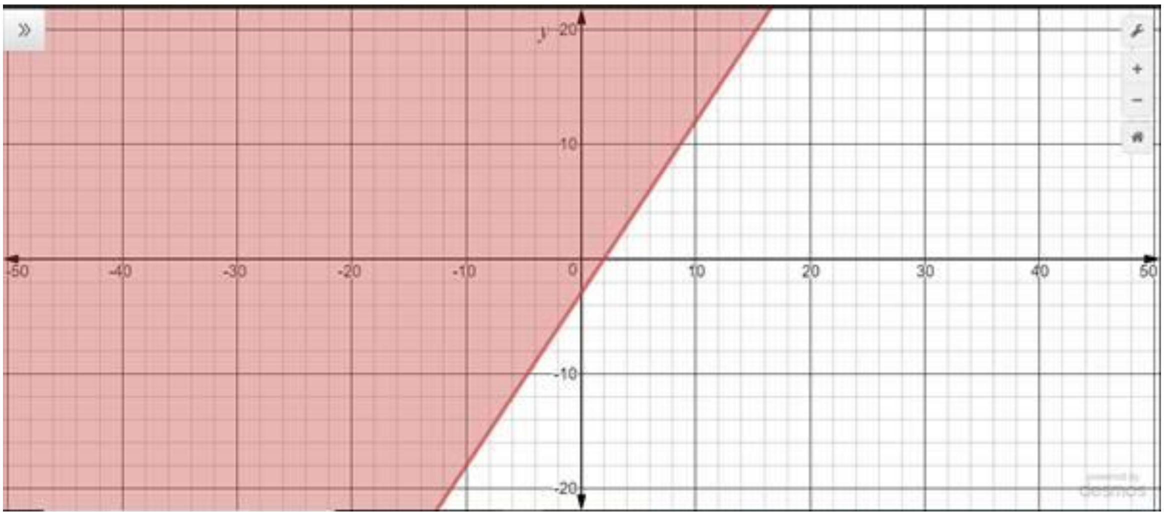
Select a point (0,0)

⇒ - 3 × (0) + 2 × (0) ≥ – 6

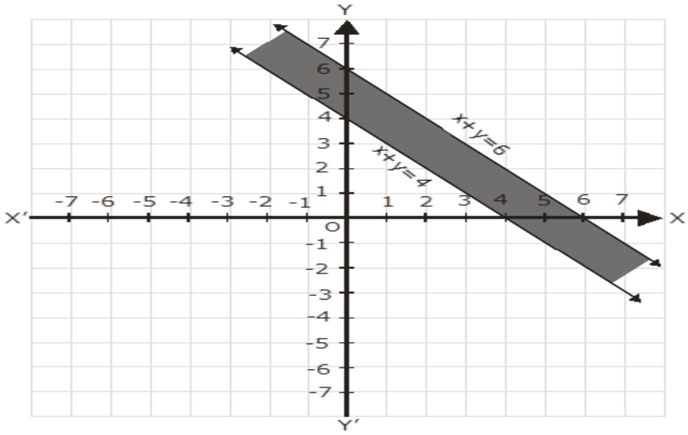
⇒ 0 ≥ – 6 (this is true)

∴ Solution region of the given inequality is above the line – 3x + 2y ≥ – 6. (That is origin is included in the region)

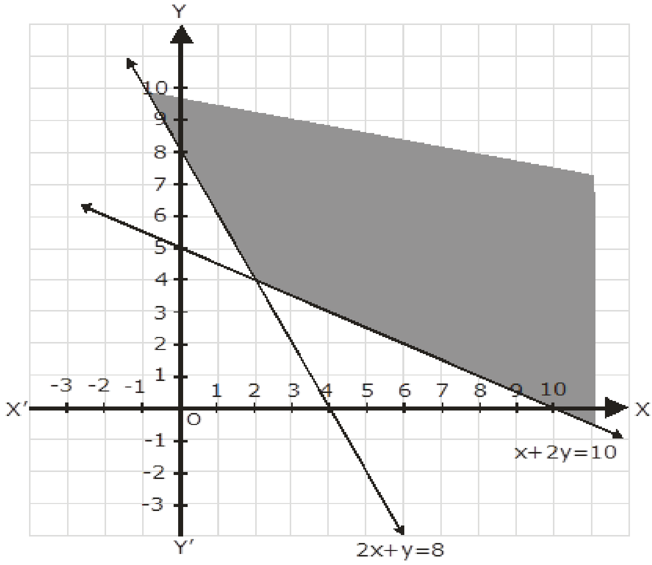
The graph is as follows:



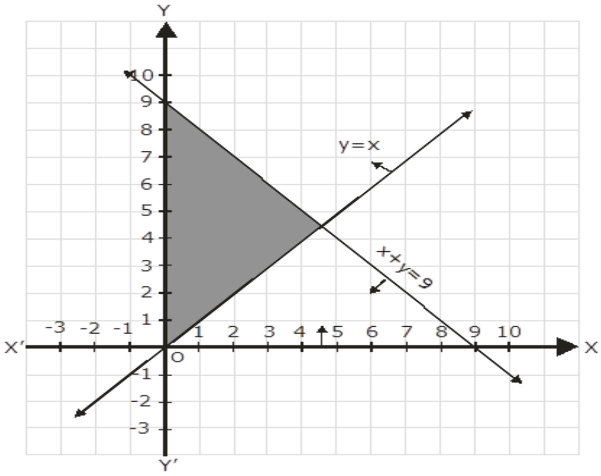
A.6.



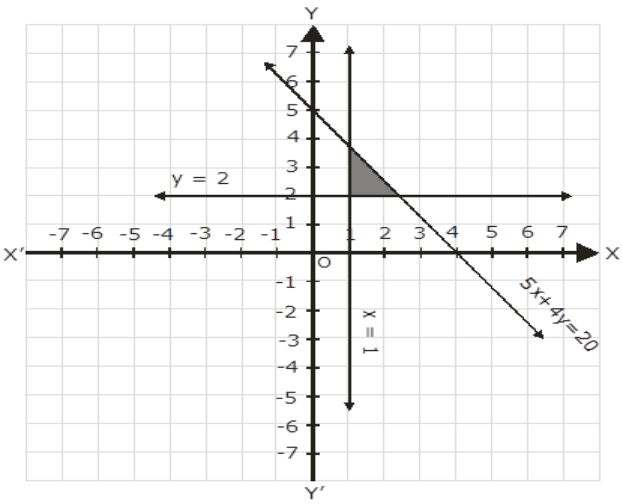
A.7.



A.8.



A.9.



A.10. The given inequality is $3x + 4y \leq 60$,

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 15 and x = 20

The required points are (0, 15) and (20, 0)

Checking if the origin lies in the required solution area (0,0)

$0 \leq 60$, this is true.

Hence the origin would lie in the solution area of the line`s graph.

The required solution area would be on the left of the line`s graph.

x +3y ≤ 30,

Putting value of x = 0 and y = 0 in equation one by one, we get value of

y = 10 and x = 30

The required points are (0, 10) and (30, 0)

Checking for the origin (0, 0)

0 ≤ 30, this is true.

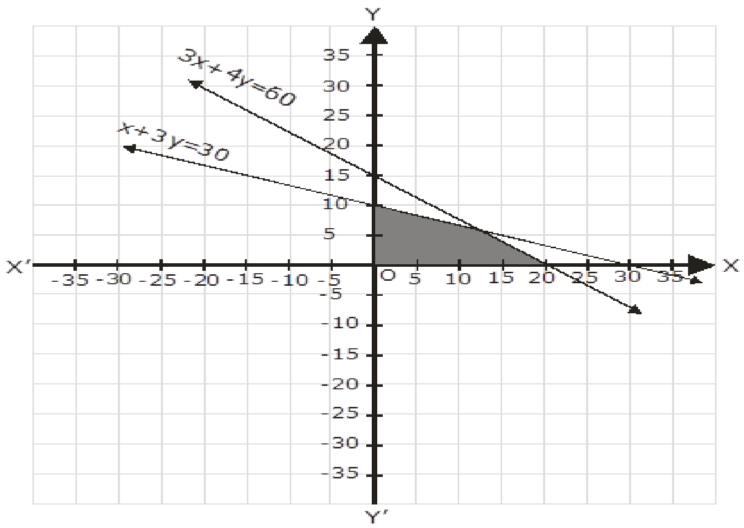
Hence the origin lies in the solution area which is given by the left side of the line`s graph.

x ≥ 0,

y ≥ 0,

The given inequalities imply the solution lies in the first quadrant only.

Hence the solution of the inequalities is given by the shaded region in the graph.



3.8 References/ Suggested Readings

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2. Dowling E.T.: Mathematics for Economics; Sihahum Series, McGraw Hill. London.
3. Kapoor, V. K.: Business Mathematics: Sultan, Chan & Sons, Delhi.
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Lesson. 4 Linear Programming

Course Name: Business Mathematics **Course Code:** BCOM 105

Semester-II **Author:** Dr Vizender Singh

Structure:

4.0 Learning Objectives

The following are learning objective of Linear Programming Problems

- To determine the solution to a linear problem.
- To minimize or maximize the function such as such profit or cost functions.
- To analyze numerous economic, social, military and industrial problem.
- To improves quality of decision: A better quality can be obtained with the system by making use of linear programming.
- To unify results from disparate areas of mechanism design.
- To find the more flexible system, a wide range of problems can be solved easily.

4.1 Introduction

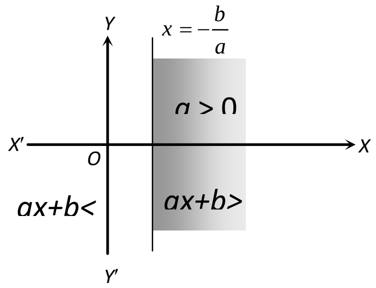
'Linear Programming' is a scientific tool to handle optimization problems. Here, we shall learn about some basic concepts of linear programming problems in two variables, their applications, advantages, limitations, formulation and graphical method of solution.

4.2 Linear Programming

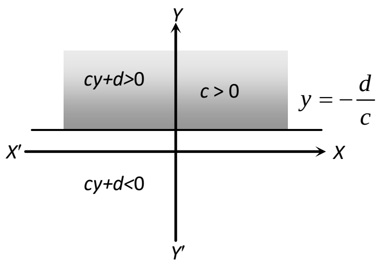
4.2.1 Linear inequations

(1) **Graph of Linear Inequations**

(i) **Linear inequation in one variable:** $ax+b>0$, $ax+b<0$, $cy+d>0$ etc., are called linear inequations in one variable. Graph of these inequations can be drawn as follows:

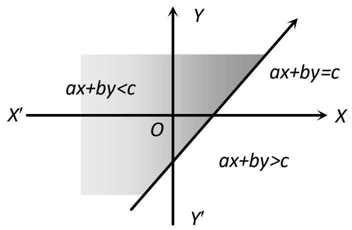


The graph of $ax+b>0$ and $ax+b<0$ are obtained by dividing xy-plane in two semi-planes by the line $x=-\frac{b}{a}$ (which is parallel to y-axis). Similarly for $cy+d>0$ and $cy+d<0$.



(ii) **Linear Inequation in two variables:** General form of these inequations are $ax+by>c, ax+by<c$. If any ordered pair (x_1, y_1) satisfies an inequation, then it is said to be a solution of the inequation.

The graph of these inequations is given below (for c > 0) :



Working Rule: To draw the graph of an inequation, following procedure is followed :

(i) Write the equation $ax+by=c$ in place of $ax+by<c$ and $ax+by>c$.

(ii) Make a table for the solutions of $ax+by=c$.

(iii) Now draw a line with the help of these points. This is the graph of the line $ax+by=c$.

(iv) If the inequation is > or <, then the points lying on this line is not considered and line is drawn dotted or discontinuous.

(v) If the ineuqation is ≥ or ≤, then the points lying on the line is considered and line is drawn bold or continuous.

(vi) This line divides the plane *XOY* in two region.

To Find the region that satisfies the inequation, we apply the following rules:

(a) Take an arbitrary point which will be in either region.

(b) If it satisfies the given inequation, then the required region will be the region in which the arbitrary point is located.

(c) If it does not satisfy the inequation, then the other region is the required region.

(d) Draw the lines in the required region or make it shaded.

(2) **Simultaneous linear inequations in two variables:** Since the solution set of a system of simultaneous linear inequations is the set of all points in two dimensional space which satisfy all the inequations simultaneously. Therefore to find the solution set we find the region of the plane common to all the portions comprising the solution set of given inequations. In case there is no region common to all the solutions of the given inequations, we say that the solution set is **void** or **empty**.

(3) **Feasible region:** The limited (bounded) region of the graph made by two inequations is called feasible region. All the points in feasible region constitute the solution of a system of inequations. The feasible solution of a L.P.P. belongs to only first quadrant. If feasible region is empty then there is no solution for the problem.

4.2.2 Terms of linear programming

The term programming means planning and refers to a process of determining a particular program.

(1) **Objective Function :** The linear function which is to be optimized (maximized or minimized) is called objective function of the *L.P.P.*

(2) **Constraints or Restrictions :** The conditions of the problem expressed as simultaneous equations or inequations are called constraints or restrictions.

(3) **Non-negative Constraints :** Variables applied in the objective function of a linear programming problem are always non-negative. The inequations which represent such constraints are called non-negative constraints.

(4) **Basic Variables :** The m variables associated with columns of the $m \times n$ non-singular matrix which may be different from zero, are called basic variables.

(5) **Basic Solution :** A solution in which the vectors associated to m variables are linear and the remaining $(n-m)$ variables are zero, is called a basic solution. A basic solution is called a degenerate basic solution, if at least one of the basic variables is zero and basic solution is called non-degenerate, if none of the basic variables is zero.

(6) **Feasible Solution:** The set of values of the variables which satisfies the set of constraints of linear programming problem (*L.P.P*) is called a feasible solution of the *L.P.P.*

(7) **Optimal Solution :** A feasible solution for which the objective function is minimum or maximum is called optimal solution.

(8) **Iso-Profit Line :** The line drawn in geometrical area of feasible region of *L.P.P.* for which the objective function (to be maximized) remains constant at all the points lying on the line, is called iso-profit line.

If the objective function is to be minimized then these lines are called **iso-cost** lines.

(9) **Convex set :** In linear programming problems feasible solution is generally a polygon

in first quadrant. This polygon is convex. It means if two points of polygon are connected by a line, then the line must be inside the polygon. For example,

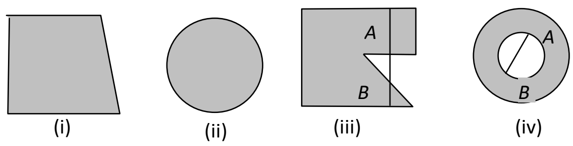


Fig. (i) and (ii) are convex set while (iii) and (iv) are not convex set.

4.2.3 Mathematical Formulation of a Linear Programming Problem

There are mainly four steps in the mathematical formulation of a linear programming problem, as mathematical model. We will discuss formulation of those problems which involve only two variables.

(1) Identify the decision variables and assign symbols x and y to them. These decision variables are those quantities whose values we wish to determine.

(2) Identify the set of constraints and express them as linear equations/inequations in terms of the decision variables. These constraints are the given conditions.

(3) Identify the objective function and express it as a linear function of decision variables. It may take the form of maximizing profit or production or minimizing cost.

(4) Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

4.2.4 Graphical Solution of Two Variable Linear Programming Problem

There are two techniques of solving an *L.P.P.* by graphical method. These are :

(1) Corner point method (2) Iso-profit or Iso-cost method

(1) **Corner Point Method**

***Working Rule*:**

(i) Formulate mathematically the *L.P.P.*

(ii) Draw graph for every constraint.

(iii) Find the feasible solution region.

(iv) Find the coordinates of the vertices of feasible solution region.

(v) Calculate the value of objective function at these vertices.

(vi) Optimal value (minimum or maximum) is the required solution.

(vii) If there is no possibility to determine the point at which the suitable solution found, then the solution of problem is unbounded.

(viii) If feasible region is empty, then there is no solution for the problem.

(ix) Nearer to the origin, the objective function is minimum and that of further from the origin, the objective function is maximum.

(2) **Iso-Profit or Iso-Cost Method:** Various steps of the method are as follows:

(i) Find the feasible region of the *L.P.P.*

(ii) Assign a constant value Z_1 to Z and draw the corresponding line of the objective function.

(iii) Assign another value Z_2 to Z and draw the corresponding line of the objective function.

(iv) If $Z_1 < Z_2, (Z_1 > Z_2)$, then in case of maximization (minimization) move the line P_1Q_1 corresponding to Z_1 to the line P_2Q_2 corresponding to Z_2 parallel to itself as far as possible, until the farthest point within the feasible region is touched by this line. The coordinates of the point give maximum (minimum) value of the objective function.

(v) The problem with more equations/inequations can be handled easily by this method.

(vi) In case of unbounded region, it either finds an optimal solution or declares an unbounded solution. Unbounded solutions are not considered optimal solution. In real world problems, unlimited profit or loss is not possible.

4.2.5. To Find the Vertices of Simple Feasible Region without Drawing a Graph

(1) **Bounded Region:** The region surrounded by the inequations $ax+by \leq m$ and $cx+dy \leq n$ in first quadrant is called bounded region. It is of the form of triangle or quadrilateral. Change these inequations into equations, then by putting $x=0$ and $y=0$, we get the solution. Also by solving the equations, we get the vertices of bounded region.

The maximum value of objective function lies at one vertex in limited region.

(2) **Unbounded Region:** The region surrounded by the inequations $ax+by \geq m$ and $cx+dy \geq n$ in first quadrant, is called unbounded region.

Change the inequation in equations and solve for $x=0$ and $y=0$. Thus we get the vertices of feasible region.

The minimum value of objective function lies at one vertex in unbounded region but there is no existence of maximum value.

4.2.6 Advantages and Limitations of L.P.P.

(1) **Advantages:** Linear programming is used to minimize the cost of production for maximum output. In short, with the help of linear programming models, a decision maker can most efficiently and effectively employ his production factor and limited resources to get maximum profit at minimum cost.

(2) **Limitations:** (i) The linear programming can be applied only when the objective function and all the constraints can be expressed in terms of linear equations/inequations.

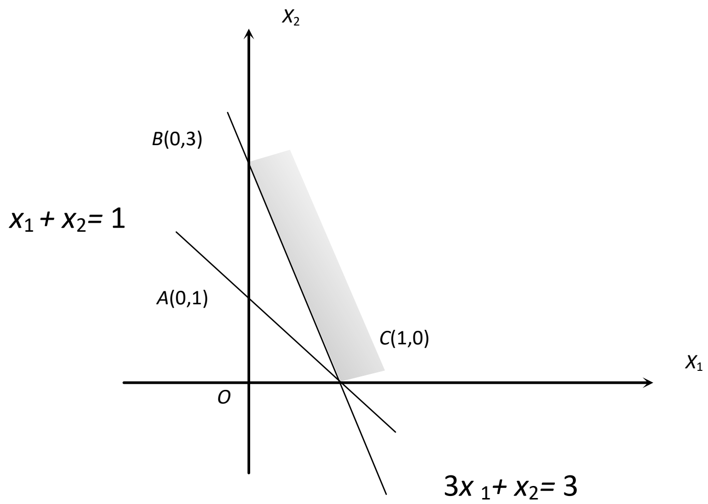
(ii) Linear programming techniques provide solutions only when all the elements related to a problem can be quantified.

(iii) The coefficients in the objective function and in the constraints must be known with certainty and should remain unchanged during the period of study.

(iv) Linear programming technique may give fractional valued answer which is not desirable in some problems.

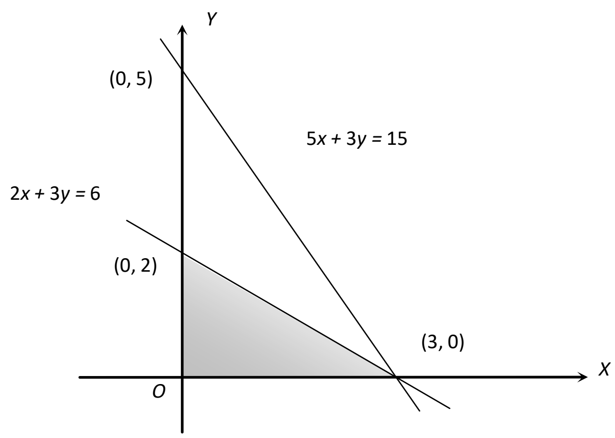
Example 4.2.7 Find the number of feasible regions for the constraint of a linear optimizing function $z = x_1 + x_2$, given by $x_1 + x_2 \le 1, 3x_1 + x_2 \ge 3$ and $x_1, x_2 \ge 0$

Solution. Clearly from graph there is no feasible region.



Example 4.2.8 Find the points which are the vertices of the positive region bounded by the inequalities $2x + 3y \le 6$, $5x + 3y \le 15$ and $x, y \ge 0$.

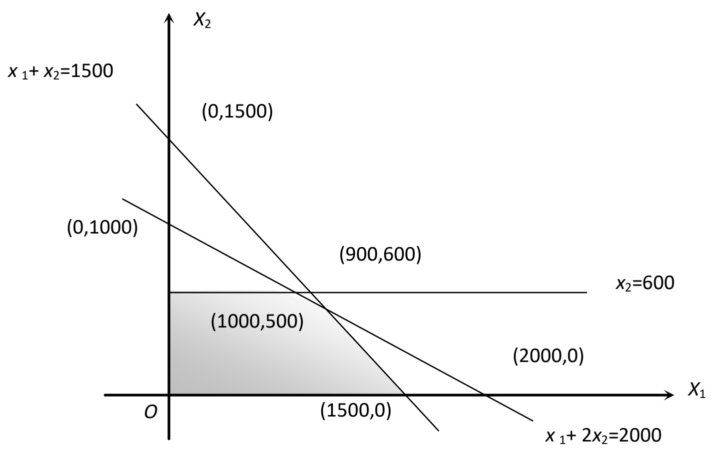
Solution. Drawing the graph, we have, the feasible region is the shaded are and



clearly (0, 2); (0, 0) and (3, 0) all are vertices of feasible region.

Example 4.2.9 For the constraints of a L.P.P. problem given by $x_1 + 2x_2 \le 2000$, $x_1 + x_2 \le 1500$, $x_2 \le 600$ and $x_1, x_2 \ge 0$, find the points which lies in the positive bounded region.

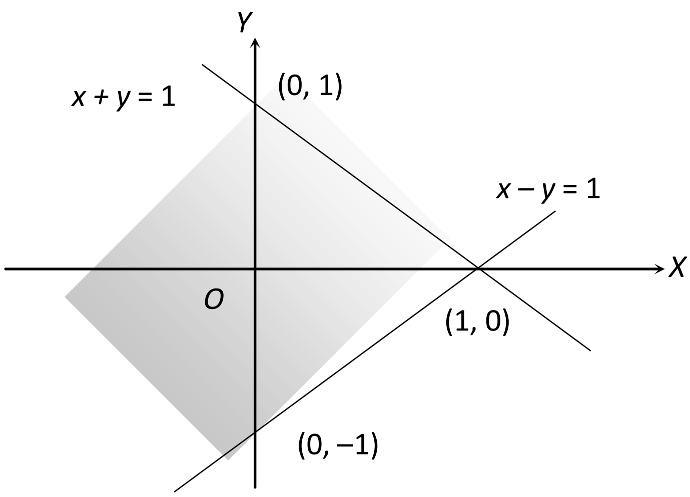
Solution. Drawing the graph, we have, the feasible region is the shaded are and



clearly the vertices, (0, 0), (0, 1000); (1000, 500) and (1500, 0) all lies in the positive bounded region.

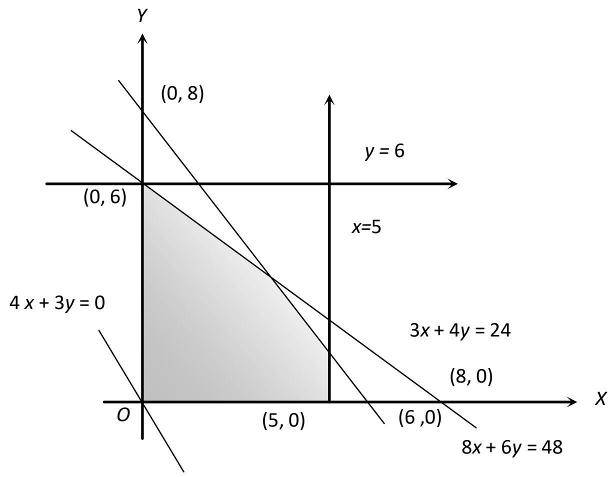
Example 4.2.10 Find the quadrant in which, the bounded region for inequations $x + y \le 1$ and $x - y \le 1$ is situated.

Solution. As shown in graph drawn for $x + y = 1$ and $x - y = 1$ the origin included in the area. Hence the bounded region situated in all four quadrant.



Example 4.2.11 Find the number of points at which the objective function $z = 4x + 3y$ can be maximized subjected to the constraints $3x + 4y \leq 24$, $8x + 6y \leq 48$, $x \leq 5, y \leq 6;\ x, y \geq 0$.

Solutiojn. Obviously, the optimal solution is found on the line which is parallel to 'isoprofit line'. Hence it has infinite number of solution.



Example 4.2.12 A wholesale merchant wants to start the business of cereal with *Rs*. 24000. Wheat is *Rs*. 400 per *quintal* and rice is Rs. 600 per *quintal*. He has capacity to store 200 *quintal* cereal. He earns the profit *Rs*. 25 per *quintal* on wheat and *Rs*. 40 per *quintal* on rice. If he stores *x quintal* rice and *y quintal* wheat, then for maximum profit find the objective function.

Solution. For maximum profit, $z = 40x + 25y$.

Example 4.2.13 Mohan wants to invest the total amount of *Rs*. 15,000 in saving certificates and national saving bonds. According to rules, he has to invest at least *Rs*. 2000 in saving certificates and *Rs*. 2500 in national saving bonds. The interest rate is 8% on saving certificate and 10% on national saving bonds per annum. He invest *Rs*. *x* in saving certificates and *Rs*. *y* in national saving bonds. Then write the objective function for this problem.

Solution. Objective function is given by profit function

$$\text{z} = x \cdot \frac{8}{100} + y \times \frac{10}{100} = 0.08x + 0.10y \,.$$

Example 4.2.14 A firm produces two types of products *A* and *B*. The profit on both is *Rs*. 2 per item. Every product requires processing on machines M_1 and M_2. For *A*, machines M_1 and M_2 takes 1 *minute* and 2 *minute* respectively and for *B*, machines M_1 and M_2 takes the time 1 *minute* each. The machines M_1, M_2 are not available more than 8 *hours* and 10 *hours*, any of day, respectively. If the products made *x* of *A* and *y* of *B*, then find the linear constraints for the L.P.P. except $x \geq 0, y \geq 0$.

Solution. Obviously $x + y \leq (8 \times 60 = 480)$ and $2x + y \leq (10 \times 60 = 600)$.

Example 4.2.15 In a test of Mathematics, there is two types of questions to be answered–short answered and long answered. The relevant data is given below

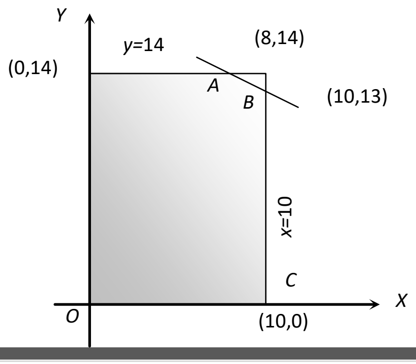
Type of questions	Time taken to solve	Marks	Number of questions
Short answered questions	5 Minutes	3	10
Long answered questions	10 Minutes	5	14

The total marks is 100. Students can solve all the questions. To secure maximum marks, a student solves *x* short answered and *y* long answered questions in three hours, then find the linear constraints except $x \geq 0, y \geq 0$.

Also, find vertices of a feasible region and maximum value of objective function.

Solution. Obviously linear constraints except are $x \leq 10, y \leq 14$ and $5x + 10y \leq 180$.

By drawing graph the required feasible region is given by $ABCD$, and vertices are (8, 14), (10, 13), (10, 0) and (0, 14)



And maximum value of objective function $Max\, z = 3(10) + 5(13) = 95$.

Example 4.2.16 A factory produces two products *A* and *B*. In the manufacturing of product A, the machine and the carpenter requires 3 *hour* each and in manufacturing of product *B*, the machine and carpenter requires 5 *hour* and 3 *hour* respectively. The machine and carpenter work at most 80 *hour* and 50 *hour* per week respectively. The profit on *A* and *B* is *Rs*. 6 and 8 respectively. If profit is maximum by manufacturing *x* and *y* units of *A* and *B* type product respectively, then for the function $6x + 8y$, write the constraints of problem.

Solution. The constraints of problem are $x, y \geq 0, 3x + 5y \leq 80, 3x + 3y \leq 50$.

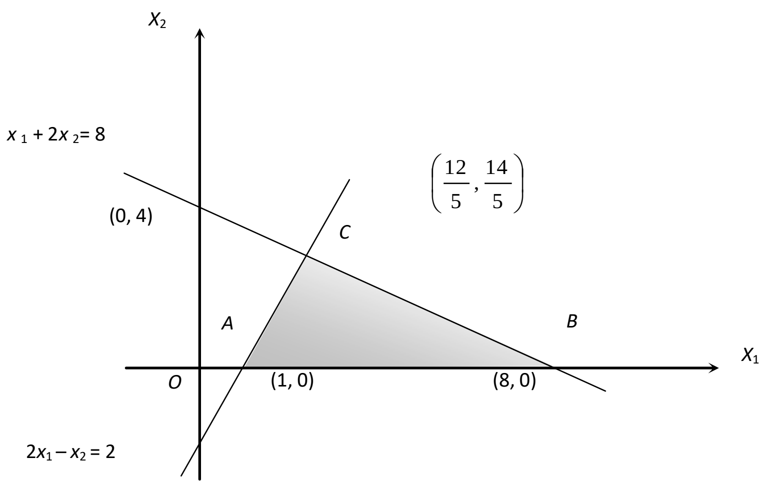
Example 4.2.17 A shopkeeper wants to purchase two articles *A* and *B* of cost price *Rs*. 4 and 3 respectively. He thought that he may earn 30 *paise* by selling article *A* and 10 *paise* by selling article *B*. He has not to purchase total article worth more than *Rs*. 24. If he purchases the number of articles of *A* and *B*, *x* and *y* respectively, then find the linear constraints. Also write the profit line.

Solution. The constraints of problem are $x \geq 0,\ y \geq 0,\ 4x + 3y \leq 24$ and the profit line is

$$4x + 3y = 24$$

Example 4.2.18 For the L.P. problem $Max\, z = 3x_1 + 2x_2$ such that $2x_1 - x_2 \geq 2$, $x_1 + 2x_2 \leq 8$ and $x_1, x_2 \geq 0$, find z.

Solution. By Changing the inequalities into equations and drawing the graph of lines, we get the required feasible region.



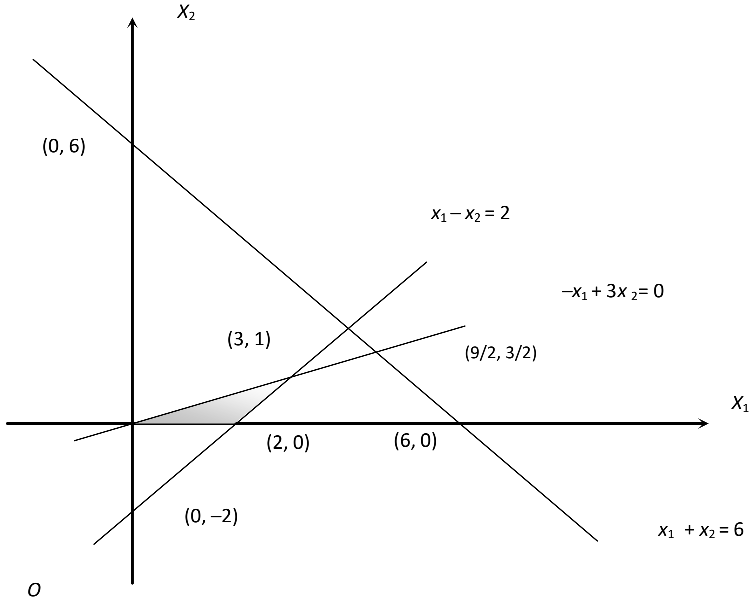
It is a bounded region, bounded by the vertices $A(1,0), B(8,0)$ and $C\left(\frac{12}{5}, \frac{14}{5}\right)$.

Now by evaluation of the objective function for the vertices of feasible region it is found to be maximum at (8,0). Hence the solution is $z = 3\times 8 + 0\times 2 = 24$.

Example 4.2.19 For the L.P. problem Min $z = -x_1 + 2x_2$ such that $-x_1 + 3x_2 \le 0,$ $x_1 + x_2 \le 6,$

$x_1 - x_2 \le 2$ and $x_1, x_2 \ge 0$, the find x_1.

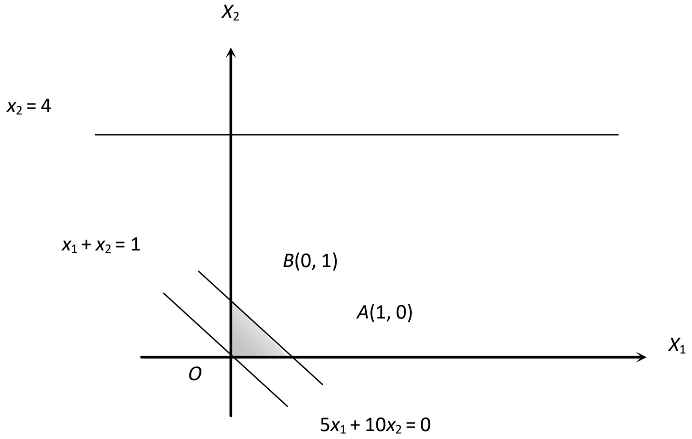
Solution. Here $(3,1),(2,0)$ are vertices of *Min z* for (2, 0)



Hence $x_1 = 2$.

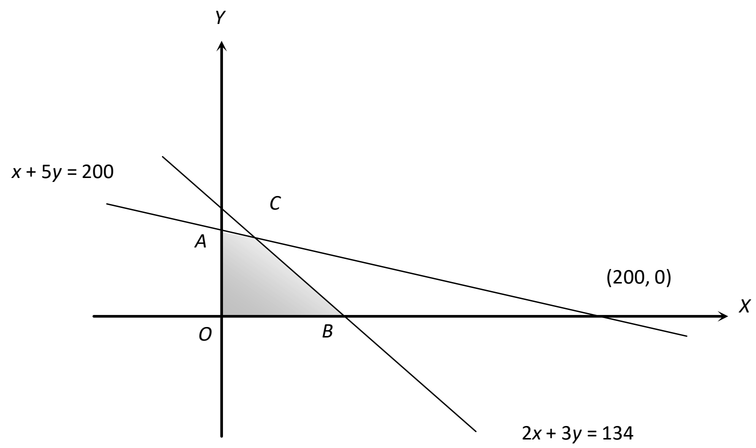
Example 4.2.20 For the L.P. problem $Min\, z = x_1 + x_2$ such that $5x_1 + 10x_2 \le 0,$ $x_1 + x_2 \ge 1,$ $x_2 \le 4$ and $x_1, x_2 \ge 0$, find the number of solutions for the problem.

Solution. As there may be infinite values of x_1 and x_2 on line $x_1 + x_2 = 1$.



Example 4.2.21 On maximizing $z = 4x + 9y$ subject to $x + 5y \le 200$, $2x + 3y \le 134$ and $x, y \ge 0$, find z.

Solution. Here, $A = (0,40), B = (67,0)$ and $C = (10,38)$



Maximum for C *i.e.*, $z = 40 + 342 = 382$.

Example 4.2.22 For the L.P. problem $Min\, z = 2x + y$ subject to $5x + 10y \le 50$, $x + y \ge 1$, $y \le 4$ and $x, y \ge 0$, Find z.

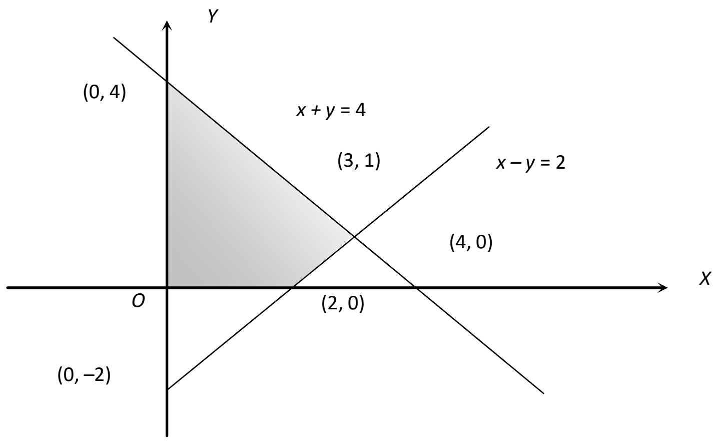
Solution. After drawing a graph, we get the vertices of feasible region are (1, 0), (10, 0), (2, 4), (0, 4) and (0, 1).

Thus minimum value of objective function is at (0, 1)

Hence $z = 0 \times 2 + 1 \times 1 = 1$.

Example 4.2.23 Find the solution of a problem to maximize the objective function $z = x + 2y$ under the constraints $x - y \le 2$, $x + y \le 4$ and $x, y \ge 0$.

Solution. Here $z = x + 2y$



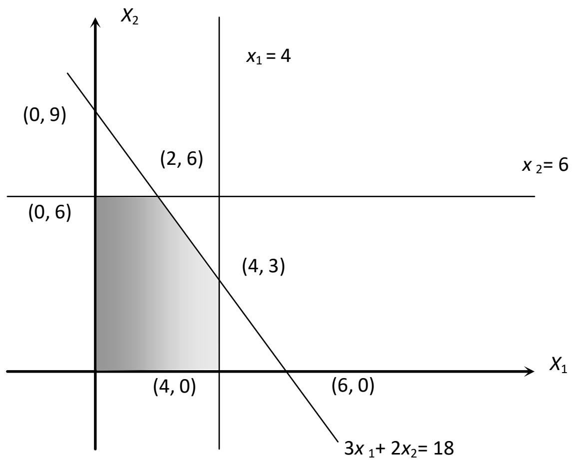
Max $z = 0 + 4(2) = 8$.

Example 4.2.24 By graphical method, find the the solution of linear programming problem

Maximize $z = 3x_1 + 5x_2$

Subject to $3x_1 + 2x_2 \le 18$, $x_1 \le 4$, $x_2 \le 6$, $x_1 \ge 0$, $x_2 \ge 0$.

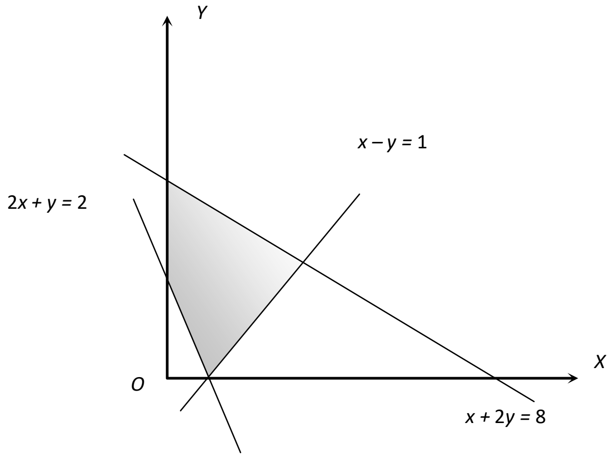
Solution. By drawing the graph, we have



Here feasible region has vertices (0, 0); (4, 0); (4, 3); (2, 6) and (0, 6).

$\therefore$ *Max z* at $(2,6) = 3(2)+5(6) = 36$.

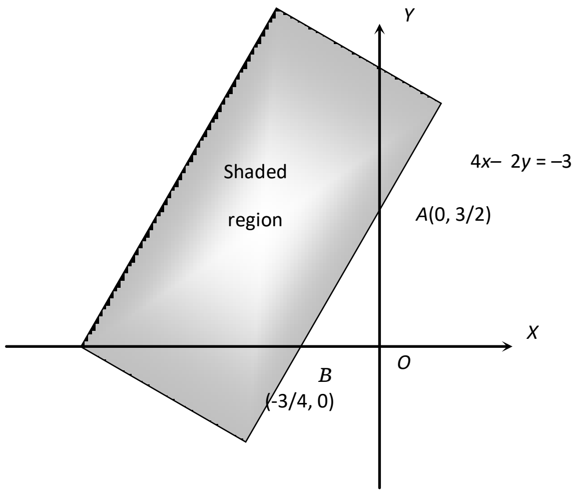
Example 4.2.25 For the following shaded area, find the linear constraints except $x \geq 0$ and $y \geq 0$.



Solution. To test the origin for $2x + y = 2, x - y = 1$ and $x + 2y = 8$ in reference to shaded area, $0+0<2$ is true for $2x + y = 2$. So for the region does not include origin (0, 0), $2x + y \geq 2$. Again for $x - y = 1, 0 - 0 < 1$, $\therefore \ x - y \leq 1$

Similarly for $x + 2y = 8, 0 + 0 < 8$; $\therefore \ x + 2y \leq 8$.

Example 4.2.26 Write the inequation representing the shaded region



Solution. Origin is not present in given shaded area, so $4x - 2y \le -3$ satisfy this condition.

Example 4.2.27 A Firm makes pents and shirts. A shirt takes 2 hour on machine and 3 hour of man labour while a pent takes 3 hour on machine and 2 hour of man labour. In a week there are 70 hour machine and 75 hour of man labour available. If the firm determine to make *x* shirts and *y* pents per week, then for this L.P.P. write the linear constraints.

Solution. Here

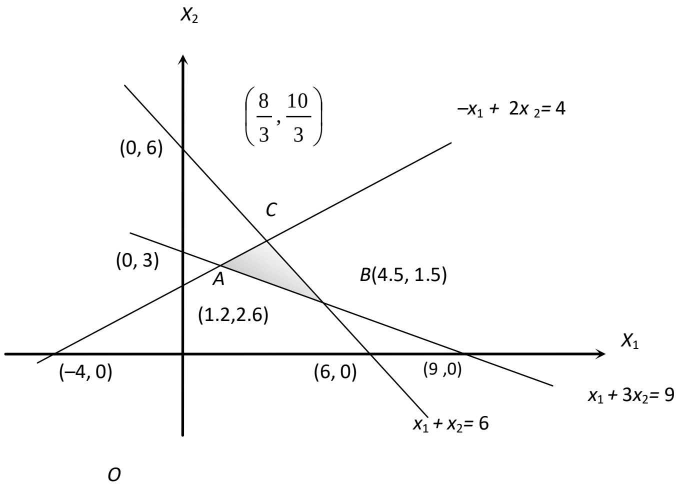
Type of items	Working time on machine	Man labour
Shirt (x) Pent (y)	2 hours 3 hours	3 hours 2 hours
Availability	70 hours	75 hours

Linear constraints are $2x + 3y \le 70, 3x + 2y \le 75$.

Example 4.2.28 For the L.P. problem $Min\ z = 2x_1 + 3x_2$ such that $-x_1 + 2x_2 \le 4$,

$x_1 + x_2 \le 6,\ x_1 + 3x_2 \ge 9$ and $x_1, x_2 \ge 0$. Find variables and objective function z.

Solution. The graph of linear programming problem is as given below



Hence the required feasible region is given by the graph whose vertices are $A(1.2, 2.6), B(4.5, 1.5)$ and $C\left(\frac{8}{3}, \frac{10}{3}\right)$.

Thus objective function is minimum at $A(1.2, 2.6)$

So $x_1 = 1.2, x_2 = 2.6$ and $z = 2\times 1.2 + 3\times 2.6 = 10.2$.

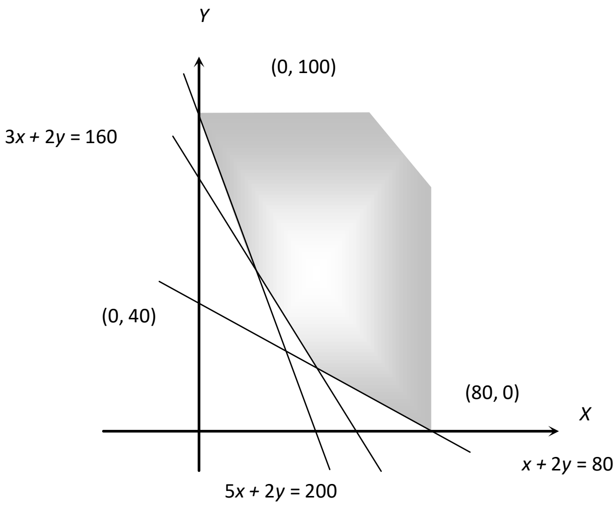
Example 4.2.29 A company manufactures two types of products *A* and *B*. The storage capacity of its godown is 100 units. Total investment amount is *Rs*. 30,000. The cost price of *A* and *B* are *Rs*. 400 and *Rs*. 900 respectively. If all the products have sold and per unit profit is *Rs*. 100 and *Rs*. 120 through *A* and *B* respectively. If *x* units of *A* and *y* units of *B* be produced, then find the two linear constraints and iso-profit line.

Solution. Here $x + y \le 100$, $400x + 900y \le 30000$ or

$4x + 9y \le 300$ and $100x + 120y = c$.

Example 4.2.30 Find the the maximum value of $z = 4x + 3y$ subject to the constraints $3x + 2y \ge 160$, $5x + 2y \ge 200$, $x + 2y \ge 80$; $x, y \ge 0$

Solutions. Obviously, it is unbounded. Therefore its maximum value does not exist.



Example 4.2.31 Minimize $z = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij}$

Subject to : $\sum_{j=1}^{n} x_{ij} \le a_i$, $i = 1, \ldots\ldots, m$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1,......,\ n$$

is a (L.P.P.) with number of constraints as $m+n$.

Solution. Here

$i = 1, x_{11} + x_{12} + x_{13} + ... + x_{1n}$

$i = 2, x_{21} + x_{22} + x_{23} + ... + x_{2n}$

$i = 3, x_{31} + x_{32} + x_{33} + ... + x_{3n}$

....

$i = m, x_{m1} + x_{m2} + x_{m3} + ... x_{mn} \rightarrow$ constraints

Condition (ii),

$j = 1, x_{11} + x_{21} + x_{31} + ... + x_{m1}$

$j = 2, x_{12} + x_{22} + x_{32} + ... + x_{m1}$

....

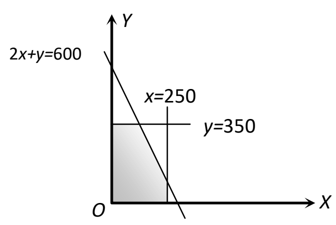
$j = n, x_{1n} + x_{2n} + x_{3n} + ... + x_{mn} \rightarrow n$ constraints

$\therefore$ Total constraints = $m+n$.

4.3 Check Your Progress

Answer the following objective type questions.

Q.1. For the following feasible region, the linear constraints except $x \geq 0$ and $y \geq 0$, are



(a) $x \geq 250$, $y \leq 350$, $2x + y = 600$

(b) $x \leq 250$, $y \leq 350$, $2x + y = 600$

(c) $x \leq 250$, $y \leq 350$, $2x + y \geq 600$

(d) $x \leq 250$, $y \leq 350$, $2x + y \leq 600$

Q.2. Let X_1 and X_2 are optimal solutions of a L.P.P., then

(a) $X = \lambda X_1 + (1-\lambda)X_2, \lambda \in R$ is also an optimal solution

(b) $X = \lambda X_1 + (1-\lambda)X_2, 0 \leq \lambda \leq 1$ gives an optimal solution

(c) $X = \lambda X_1 + (1+\lambda)X_2, 0 \leq \lambda \leq 1$ gives an optimal solution

(d) $X = \lambda X_1 + (1+\lambda)X_2, \lambda \in R$ gives an optimal solution

Q.3. The points which provides the solution to the linear programming problem: $Max(2x + 3y)$ subject to constraints: $x \geq 0, y \geq 0, \; 2x + 2y \leq 9, \; 2x + y \leq 7$, $x + 2y \leq 8$, is

(a) (3, 2.5) (b) (2, 3.5)

(c) (2, 2.5) (d) (1, 3.5)

Q.4. Two tailors *A* and *B* earns *Rs.* 15 and *Rs.* 20 per day respectively. A can make 6 shirts and 4 pents in a day while *B* can make 10 shirts and 3 pents. To spend minimum on 60 shirts and 40 pents, *A* and *B* working *x* and *y* days respectively. Then linear constraints except $x \geq 0, y \geq 0$, are and objective function are respectively

(a) $15x + 20y \geq 0, 60x + 40y \geq 0$, $z = 4x + 3y$

(b) $15x + 20y \geq 0, 6x + 10y = 10$, $z = 60x + 60y$

(c) $6x + 10y \geq 60, 4x + 3y \geq 40$, $z = 60x + 40y$

(d) $6x + 10y \leq 60, 4x + 3y \leq 40$, $z = 15x + 20y$

Q.5. A company manufactures two types of telephone sets *A* and *B*. The *A* type telephone set requires 2 hour and *B* type telephone requires 4 hour to make. The company has 800 work hour per day. 300

telephone can pack in a day. The selling prices of *A* and *B* type telephones are *Rs*. 300 and 400 respectively. For maximum profits company produces *x* telephones of *A* type and *y* telephones of *B* types. Then except $x \geq 0$ and $y \geq 0$, linear constraints and the probable region of the L.P.P is of the type

(a) $x + 2y \leq 400; x + y \leq 300$; $Max\, z = 300x + 400y$, bounded

(b) $2x + y \leq 400; x + y \geq 300$; $Max\, z = 400x + 300y$, unbounded

(c) $2x + y \geq 400; x + y \geq 300$; $Max\, z = 300x + 400y$, parallelogram

(d) $x + 2y \leq 400; x + y \geq 300$; $Max\, z = 300x + 400y$, square

Q.6. We have to purchase two articles *A* and *B* of cost *Rs*. 45 and *Rs*. 25 respectively. I can purchase total article maximum of *Rs*. 1000. After selling the articles *A* and *B*, the profit per unit is *Rs*. 5 and 3 respectively. If I purchase the *x* and *y* numbers of articles *A* and *B* respectively, then the mathematical formulation of problem is

(a) $x \geq 0, y \geq 0,\ 45x + 25y \geq 1000,\ 5x + 3y = c$

(b) $x \geq 0, y \geq 0,\ 45x + 25y \leq 1000,\ 5x + 3y = c$

(c) $x \geq 0, y \geq 0,\ 45x + 25y \leq 1000,\ 3x + 5y = c$

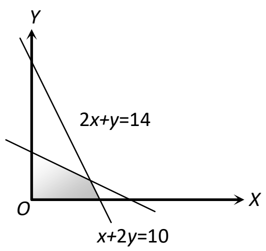
(d) None of these

Q.7. For the L.P. problem $Max\, z = 3x + 2y$ subject to $x + y \geq 1,\ y - 5x \leq 0,\ x - y \geq -1,\ x + y \leq 6,\ x \leq 3$ and $x, y \geq 0$

(a) $x = 3$ (b) $y = 3$

(c) $z = 15$ (d) All the above

Q.8. The maximum value of objective function $c = 2x + 3y$ in the given feasible region, is



(a) 29

(b) 18

(c) 14

(d) 15

Q.9. The maximum value of $4x + 5y$ subject to the constraints $x + y \le 20,\ x + 2y \le 35,\ x - 3y \le 12$ is

(a) 84 (b) 95

(c) 100 (d) 96

Q.10. For the following linear programming problem: minimize $z = 4x + 6y$ subject to the constraints $2x + 3y \ge 6$, $x + y \le 8,\ \ y \ge 1,\ x \ge 0$, the solution is

(a) (0, 2) and (1, 1) (b) (0, 2) and $(3/2, 1)$

(c) (0, 2) and (1, 6) (d) (0, 2) and (1, 5)

4.4 Summary

- **In some of the linear programming problems, constraints are inconsistent *i.e.* there does not exist any point which satisfies all the constraints. Such type of linear programming problems are said to have *infeasible solution*.**
- **If the constraints in a linear programming problem are changed, the problem is to be re-evaluated.**
- **The optimal value of the objective function is attained at the point, given by corner points of the feasible region.**
- **If a *L.P.P.* admits two optimal solutions, it has an infinite number of optimal solutions.**
- **If there is no possibility to determine the point at which the suitable solution can be found, then the solution of problem is unbounded.**

✍ **The maximum value of objective function lies at one vertex in limited region.**

4.5 Keywords

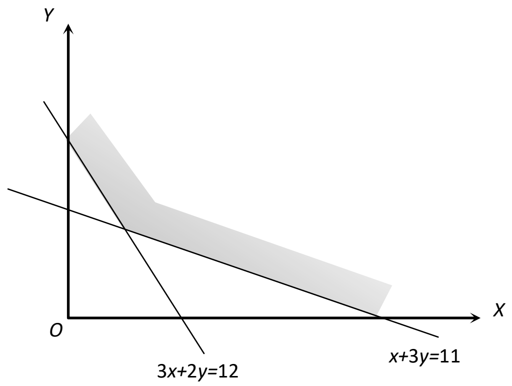
Linear equations and in equations, coordinate geometry, maxima and minima, profit and cost function.

4.6 Self-Assessment Test

Q.1. Find the vertex of common graph of inequalities $2x + y \geq 2$ and $x - y \leq 3$.

Q.2. Write the necessary condition for third quadrant region in *xy*-plane.

Q.3. For the following feasible region, find the linear constraints.



Q.4. Find the number of points at which the objective function $z = 4x + 3y$ can be maximized subjected to the constraints $3x + 4y \leq 24$, $8x + 6y \leq 48$, $x \leq 5, y \leq 6;\ x, y \geq 0$.

Q.5. Write the type of feasible region on which constraints

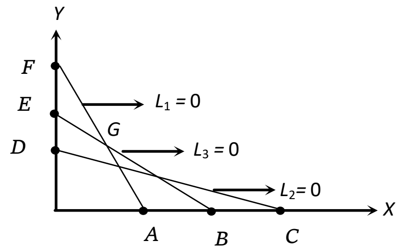
$$-x_1 + x_2 \leq 1$$

$$-x_1 + 3x_2 \leq 9$$

$x_1, x_2 \geq 0$ are defined.

Q.6. The quadrant in which the graph of inequations $x \leq y$ and $y \leq x + 3$ is located.

Q.7. The feasible region for the following constraints $L_1 \le 0, L_2 \ge 0, L_3 = 0, x \ge 0, y \ge 0$ in the diagram shown is



Q.8. The sum of two positive integers is at most 5. The difference between two times of second number and first number is at most 4. If the first number is x and second number$y$, then for maximizing the product of these two numbers, write the mathematical formulation og L.P.P.

Q.9. Find the number of solutions for the L.P. problem *Min* $z = x_1 + x_2$ such that

$5x_1 + 10x_2 \le 0,\ x_1 + x_2 \ge 1,\ x_2 \le 4$ and . $x_1, x_2 \ge 0$

Q.10. Find the maximum value of $P = 6x + 8y$ subject to constraints $2x + y \le 30, x + 2y \le 24$ and $x \ge 0, y \ge 0$.

Q.11. Find the maximum value of $P = x + 3y$ such that $2x + y \le 20$, $x + 2y \le 20$, $x \ge 0, y \ge 0$.

Q.12. Find the coordinate ofpoint at which the maximum value of $(x + y)$ subject to the constraints $2x + 5y \le 100$, $\frac{x}{25} + \frac{y}{49} \le 1$, $x, y \ge 0$ is obtained.

Q.13. Write the minimum value of $z = 2x_1 + 3x_2$ subject to the constraints

$2x_1 + 7x_2 \ge 22$, $x_1 + x_2 \ge 6$, $5x_1 + x_2 \ge 10$ and $x_1, x_2 \ge 0$.

Q.14. Find the co-ordinates of the point for minimum value of $z = 7x - 8y$ subject to the conditions $x + y - 20 \le 0$, $y \ge 5$, $x \ge 0$, $y \ge 0$.

Q.15. The maximum value of $\mu = 3x + 4y$, subject to the conditions

$x + y \le 40, x + 2y \le 60, x, y \ge 0$.

Answers

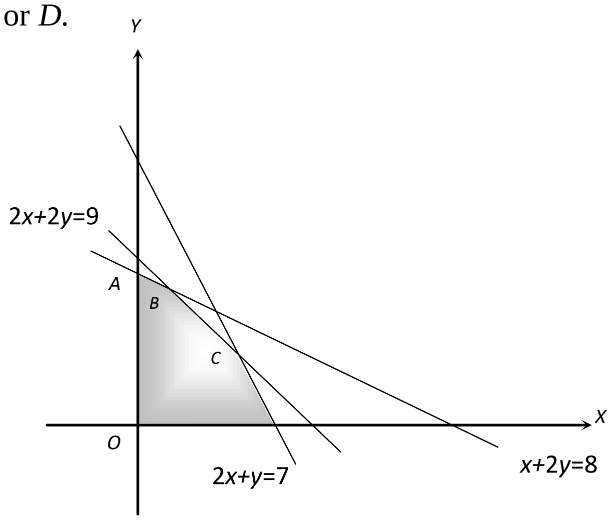
1	$\left(\frac{5}{3}, -\frac{4}{3}\right)$	2	$x < 0, y < 0$	3	$x \ge 0,\ y \ge 0,$ $x + 3y \ge 11$ $3x + 2y \ge 12,$	4	At an infinite number of points.	5	Both bounded and unbounded feasible space
6	I, II and III quadrants	7	Line segment *EG*	8	$x + y \le 5$ $2y - x \le 4,$ $x \ge 0,\ y \ge 0$	9	There are infinite solutions	10	120
11	30	12	$\left(\frac{50}{3}, \frac{40}{3}\right)$	13	14	14	(0, 20)	15	140

4.7 Answers to check your progress

1. (d) It is obvious.

2. (b) It is a fundamental concept.

3. (d) Given, $P = 2x + 3y$ Graph has been shown by given constraints and maximum value of P can be on A or B or C or D.



$P_A = P_{(0,4)} = 2(0) + 3(4) = 12$

$P_B = P_{(1,3.5)} = 2 \times 1 + 3 \times 3.5 = 12.5$

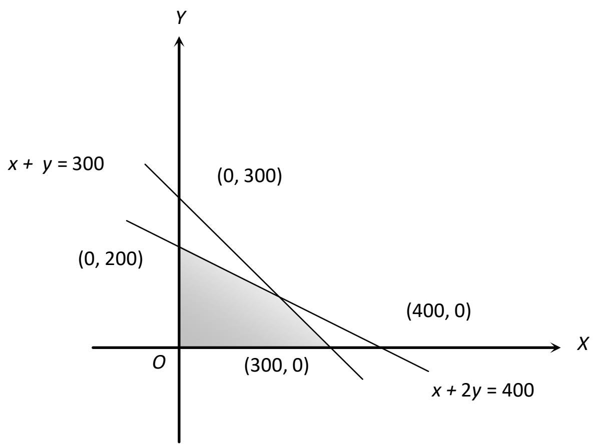
$P_C = P_{(2.5,2)} = 2 \times 2.5 + 3 \times 2 = 11$

$P_D = P_{(3.5,0)} = 2 \times 3.5 + 3 \times 0 = 7$

Obviously, at (1,3.5), $P_{\max} = 12.5$

4. (c) $6x + 10y \geq 60, 4x + 3y \geq 40$ (As minimum expenditure is concerned).

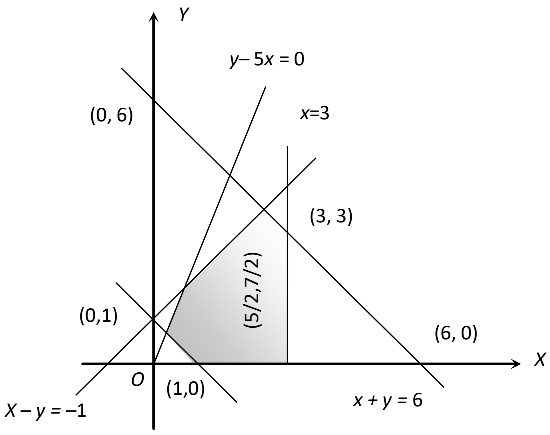
5. (a) The linear constraints are $x + 2y \leq 400, x + y \leq 300$ and $x, y \geq 0$. Also *Max* $z = 300x + 400y$.



Hence the region is bounded.

6. (b) $x \geq 0, y \geq 0,\ 45x + 25y \leq 1000, 5x + 3y = c$.

7. (d) The shaded region represents the bounded region.



Now, we get the maximum value of z at vertex (3, 3). So $Max\, z = 3(3) + 2(3) = 15$.

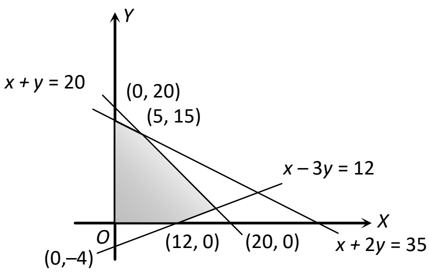
8. (b) The two vertices of given feasible region are (0, 5) and (7, 0) and third vertex can be found by solving the equations $x + 2y = 10$ and $2x + y = 14$, we get (6, 2) Now at (0, 5) $c = 2 \times 0 + 5 \times 3 = 15$, at (7, 0)

$c = 2 \times 7 + 0 \times 3 = 14$ and at (6, 2),

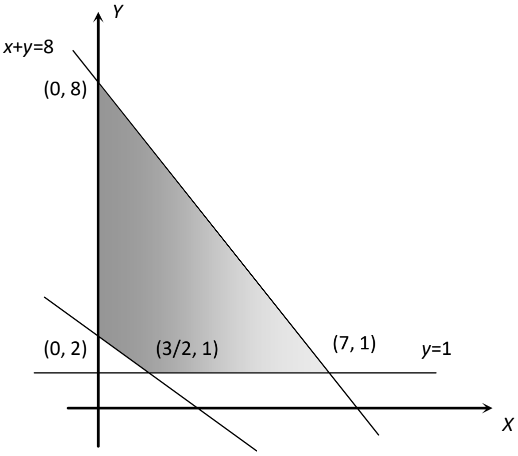
$c = 2 \times 6 + 3 \times 2 = 18$

Hence maximum value of objective function $c = 2x + 3y$ is 18 at point (6, 2).

9. (b) Obviously, $Max(4x + 5y) = 95$. It is at (5, 15).



10. (b)



Obviously, at $\left(\frac{3}{2},1\right)$ and (0, 2), $Min\ z = 4x + 6y = 12$.

4.8 References/ Suggested Readings

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